Notes for Wed 2/25

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HW for Monday 3/1

Problem session as usual Thurs 1:30-2:50
LCB 225

9.3.7 #29

11.1.6 #2c, 3, 5

Exercise 1 : X, Y normed vector spaces

$L : X \to Y$ linear $\quad \left(L(sx_1 + tx_2) = sL(x_1) + tL(x_2) \;\; \forall\, x_1, x_2 \in X,\ s, t \in \mathbb{R} \right)$

$$\|L\|_{op} = \sup_{|x| \le 1} |L(x)| = \sup_{|x|=1} |L(x)|$$

1a) L is continuous iff $\|L\|_{op} < \infty$

1b) $|L(x)| \le \|L\|_{op} |x| \quad \forall\, x \in X$

1c) $M : Y \to Z$ linear $\Rightarrow \|M \circ L\|_{op} \le \|M\|_{op} \|L\|_{op}$

Exercise 2 2a) Find $\|A\|_{op}$ for $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

2b) Is $\mathcal{A}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} .6 & .4 \\ .5 & .5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ a contraction of $\mathbb{R}^2$?

I forgot this one!

Exercise 3 $I \subset \mathbb{R}$ an interval

$f : I \to X$ Banach

f diff'ble on I and $f(x_0) = 0$

$f'(x) = 0 \;\; \forall\, x \in I$.

Prove $f(x) = 0 \;\; \forall\, x \in I$

↑ old

↓ new

Exercise ~~8~~ 4 Now let X, Y Banach (complete normed vector spaces)

Define $\mathcal{L}(X,Y) := \{ L : X \to Y \text{ linear s.t. } \|L\|_{op} < \infty \}$

note $\mathcal{L}$ is a vector space if we define $c_1 L_1 + c_2 L_2$ by $(c_1 L_1 + c_2 L_2) v = c_1 L_1(v) + c_2 L_2(v)$.

4a) Prove that $(\mathcal{L}(X,Y), \|L\|_{op})$ is a normed vectorspace (you did this before for matrices)

4b) Let $\{L_k\}$ be Cauchy in $\mathcal{L}(X,Y)$.

prove that $L(v) := \lim_{k \to \infty} L_k(v)$ exists $\forall\, v \in V$ and is a linear operator, with $\|L\|_{op} < \infty$

Then show $\{L_k\} \to L$ in $\mathcal{L}(X,Y)$, so $\mathcal{L}(X,Y)$ is complete, i.e. Banach.

Exercise ~~4~~ 5 Let $L \in \mathcal{L}(X,X)$. Examples of (4b):

5a) If $\|L\| < 1$, and $I : X \to X$ is the identity map,

prove $(I - L)^{-1} = \sum_{k=0}^{\infty} L^k$ exists, and $\|(I-L)^{-1}\|_{op} \le \dfrac{1}{1 - \|L\|_{op}}$ $\quad (L^0 = I)$

5b) $e^L := \sum_{k=0}^{\infty} \dfrac{L^k}{k!}$ exists, and

$$\|e^L\|_{op} \le e^{\|L\|_{op}}.$$