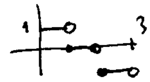
HW due Monday 1/26 lots!

9.1.4 # 4, 7, 10, 14
9.2.5 # 2, 7, 9, 14, 15

Exercise 1 Prove that the p-norm $\|\cdot\|_p$ satisfies the triangle inequality, $1 < p < \infty$.

note: Use function notation, since this includes the case of $\mathbb{R}^n$ if you identify $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$ with the integrable bounded fcn on $[0,n]$

$$f_{\vec{x}}(t) = \begin{cases} x_1 & 0 \le t < 1 \\ x_2 & 1 \le t < 2 \\ \vdots & \\ x_n & n-1 \le t < n \end{cases}$$

e.g. $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \rightsquigarrow$ 

then
$$\|x\|_p = \|f_{\vec{x}}\|_p .$$

(1a) Prove the generalized geometric-arithmetic mean inequality:

$$\boxed{\begin{matrix} \alpha, \beta \geq 0 \\ 0 < \lambda < 1 \end{matrix} \Rightarrow \alpha^{\lambda}\beta^{1-\lambda} \le \lambda\alpha + (1-\lambda)\beta \qquad \text{(equality iff } \alpha = \beta)}$$

hint: Consider, for $t > 0$, $\varphi(t) = (1-\lambda) + \lambda t - t^{\lambda}$.
Use Calculus ($\varphi'(t)$) to show that for $t > 0$ φ has a maximum value of 0, at $t = 1$.
Then, for $\beta \neq 0$, manipulate $\varphi\left(\frac{\alpha}{\beta}\right) \le 0$ to deduce result
check $\beta = 0$ separately.

(1b) Prove the generalized Cauchy-Schwarz inequality (called ~~the~~ Hölder ineq.)

$$\boxed{\int_a^b |f(t)g(t)|\,dt \le \|f\|_p \|g\|_q} \qquad \text{where } \frac{1}{p} + \frac{1}{q} = 1 \quad \left(\text{i.e. } q = \frac{p}{p-1}\right)$$

hint: First deduce Hölder in case $\|f\|_p = \|g\|_q = 1$ by deriving
$$|f(t)|\,|g(t)| \le \frac{1}{p}|f(t)|^p + \frac{1}{q}|g(t)|^q$$
from (1a), and integrating.
Derive the general case by considering $\frac{f}{\|f\|_p}$, $\frac{g}{\|g\|_q}$ and applying what you just derived.

(1c) Finally, prove $\|f+g\|_p \le \|f\|_p + \|g\|_p$ by starting with

$$\|f+g\|_p^p = \int_a^b |f(t)+g(t)|^p\,dt = \int_a^b |f(t)+g(t)|\,|f(t)+g(t)|^{p-1}\,dt$$
$$\le \int_a^b |f(t)|\,|f(t)+g(t)|^{p-1} + |g(t)|\,|f(t)+g(t)|^{p-1}\,dt$$

and Höldering these last two inequalities in the only way that would make sense!

Exercise 2 Define $\ell^2 = \{ \{x_k\}_{k \in \mathbb{N}} \}$ s.t. $\sum_{k=1}^{\infty} x_k^2 < \infty$ $\quad \left(\sum_{k=1}^{\infty} x_k^2 = \lim_{N \to \infty} \sum_{k=1}^{N} x_k^2 \right)$

(This is kind of a limit of $\mathbb{R}^n, \|\cdot\|_2$, as $n \to \infty$).

2a) Prove that $\langle v, w \rangle := \sum_{k=1}^{\infty} v_k w_k$ is a well defined inner product on ℓ^2
$(\vec{v} = \{v_k\}, \vec{w} = \{w_k\})$

2b) Prove ℓ^2 is a Hilbert space
(i.e. it's complete wrt the norm $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$)

Exercise 3 $C[0,1] := \{ f : [0,1] \to \mathbb{R}$ s.t. f continuous$\}$.

Prove $(C[0,1], \|\cdot\|_p)$ is not complete for any $1 \leq p < \infty$

Exercise 4 Let (X, d) be a metric space.

Define a relation $\equiv$ on Cauchy sequences $\{x_k\} \subset X$

$\{x_k\} \equiv \{y_k\}$ iff $\forall n \in \mathbb{N} \ \exists m \in \mathbb{N}$ s.t. $k \geq m \Rightarrow d(x_k, y_k) < 1/n$

4a) Show this is an equivalence relation

4b) Define $\tilde{X}$ = the equivalence classes of Cauchy sequences

Define $\tilde{d}([\{x_k\}], [\{y_k\}]) = \lim_{k \to \infty} d(x_k, y_k)$

Show $\tilde{d}$ exists and is well-defined

4c) Show $(\tilde{X}, \tilde{d})$ is a complete metric space, and that there is a natural "inclusion" map $i : X \to \tilde{X}$

s.t. $d(x,y) = \tilde{d}(i(x), i(y)) \ \forall x, y \in X$

Exercise 5 Let A be a subset of the metric space (X, d)

write $\mathring{A}$ for interior of A

$\bar{A}$ for closure of A

A^c for complement of A.

using these three operations, what is the maximum # of distinct sets you can create, starting with A?

Find a set for which this max # occurs. (Hint: You can find one in $\mathbb{R}$, with the usual metric)