HW sols 6 - due March 25
4530

3.3.13, 3.4.5, 3.46

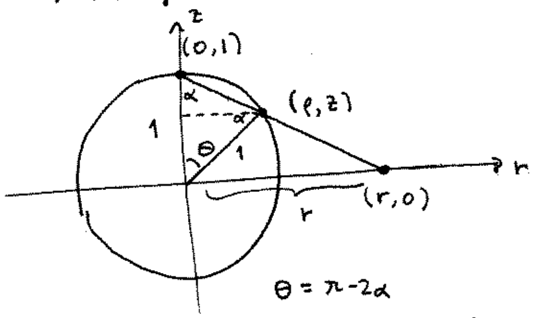
Class Exercises

1a) inverse stereographic proj. formula X

1b) show X is conformal

2) Use general Christoffel formula to rederive X_{uu} decomp.

1a)



$$\theta = \pi - 2\alpha$$

$$\rho = \sin\theta = \sin(\pi - 2\alpha) = \cos\pi \sin(-2\alpha)$$
$$= \sin 2\alpha = 2\sin\alpha\cos\alpha = 2\frac{r}{\sqrt{1+r^2}}\frac{1}{\sqrt{1+r^2}} = \frac{2r}{1+r^2}$$

$$z = \cos\theta = \cos(\pi - 2\alpha) = \cos\pi\cos 2\alpha$$
$$= -\cos 2\alpha$$
$$= -(2\cos^2\alpha - 1) = -2\frac{1}{1+r^2} + 1 = \frac{r^2-1}{r^2+1}$$

Thus, for $(u,v) \in \mathbb{R}^2$
$|(u,v)| = r$ we deduce

$$X^{-1}(u,v) = \frac{1}{1+r^2}\langle 2u, 2v, r^2-1\rangle \quad \text{as claimed.}$$

Alternate derivation:

line from N to $\langle u,v,0\rangle$

is $\alpha(t) = \langle 0,0,1\rangle + t\langle u,v,-1\rangle \qquad 0 \le t \le 1$

$= \langle tu, tv, 1-t\rangle$

We solve for t, given that $|\alpha(t)| = 1$

$$t^2(u^2+v^2) + (1-t)^2 = 1$$
$$t^2(u^2+v^2+1) - 2t + \cancel{1} = \cancel{1}$$
$$t(u^2+v^2+1) = 2$$
$$t = \frac{2}{u^2+v^2+1}$$

If you now compute $\alpha(t)$ you get $\left\langle \frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \underbrace{1 - \frac{2}{u^2+v^2+1}}_{= \frac{u^2+v^2-1}{u^2+v^2+1}} \right\rangle$

1b) The claim is that $[g_{ij}] = \begin{bmatrix} X_u \cdot X_u & X_u \cdot X_v \\ X_v \cdot X_u & X_v \cdot X_v \end{bmatrix} = \frac{4}{(1+u^2+v^2)^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

This is "just" a computation (Maple could do)

Method 1: Maple

```
> dp:=proc(X,Y)
  simplify(X[1]*Y[1]+X[2]*Y[2]+X[3]*Y[3],symbolic);
  end proc:
> X:=(u,v)->[2*u/(1+u^2+v^2),2*v/(1+u^2+v^2),(u^2+v^2-1)/(u^2+v^2+1)
  ];
```

$$X := (u,v) \to \left[2\frac{u}{1+u^2+v^2}, 2\frac{v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\right]$$

```
> Xu:=diff(X(u,v),u):
  Xv:=diff(X(u,v),v):
> dp(Xu,Xu);
  dp(Xv,Xv);
  dp(Xu,Xv);
```

$$4\frac{1}{(1+u^2+v^2)^2}$$

$$4\frac{1}{(1+u^2+v^2)^2}$$

$$0$$

```
>
```

Method 2: Check an easier special case & explain why this suffices.

At $(u,v) = (u,0)$

$$X_u(u,0) = \left\langle \frac{2(1+u^2)-4u^2}{(1+u^2)^2},\ 0,\ \frac{2u(1+u^2)-(u^2-1)2u}{(1+u^2)^2} \right\rangle$$

$$= \frac{1}{(1+u^2)^2}\langle 2(1-u^2),\ 0,\ 4u\rangle = \frac{2}{(1+u^2)^2}\langle 1-u^2, 0, 2u\rangle$$

$$X_v(u,0) = \left\langle 0,\ \frac{2}{1+u^2},\ 0\right\rangle$$

So, at $(u,v) = (u,0)$

$$X_u \cdot X_u = \frac{4}{(1+u^2)^4}\Big(\underbrace{(1-u^2)^2+4u^2}_{(1+u^2)^2}\Big) = \frac{4}{1+u^2}$$

$$X_v \cdot X_v = \frac{4}{(1+u^2)^2}$$ as claimed for $(u,0)$

$$X_u \cdot X_v = 0$$

1b) Method 2 cont'd:

Let $R = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ rotate (u_0, v_0) to $(r, 0)$ $\quad r = \sqrt{u_0^2 + v_0^2}$

Consider $X \circ R$

$$\left[(X\circ R)_u \,\middle|\, (X\circ R)_v \right] = \left[X_{,1} \,\middle|\, X_{,2} \right] \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \qquad \text{chain rule}$$

$$\Rightarrow [g_{ij}] = \begin{bmatrix} (X\circ R)_u^T \\ (X\circ R)_v^T \end{bmatrix} \left[(X\circ R)_u \,\middle|\, (X\circ R)_v \right] = R^T \begin{bmatrix} X_{,1}^T \\ X_{,2}^T \end{bmatrix} \left[X_{,1} \,\middle|\, X_{,2} \right] R$$

$$= R^T \frac{4}{(1+r^2)^2} I \; R = \left(\frac{4}{1+r^2}\right) I$$

i.e. $X \circ R$ is also conformal,
with the same conformal factor.

Finally, let $\tilde{R} = \left[\begin{array}{c|c} R^{-1} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 \; 0 & 1 \end{array} \right]$ be the corresponding inverse rotation, considered as a horiz rot in $\mathbb{R}^3$

then $\tilde{R}$ is an isometry, so preserves 1st fdl form.

Thus $X = \tilde{R} \circ X \circ R$ has desired 1st fdl form.

2) $\Gamma_{ij}^{k} = \frac{1}{2} g^{k\ell} \left(g_{\ell i,j} + g_{\ell j,i} - g_{ij,\ell} \right)$ (summation convention)

$X_{uu} = \Gamma_{11}^{1} X_u + \Gamma_{11}^{2} X_v + h_{11} U = \frac{E_u}{2E} X_u - \frac{E_v}{2G} X_v + \ell U$ (page 136, for orthogonal param.)

$\Gamma_{11}^{1} = \frac{1}{2} g^{1\ell} \left(g_{\ell 1,1} + g_{\ell 1,1} - g_{11,\ell} \right)$

summed over $\ell = 1,2$. But $[g^{ij}] = [g_{ij}]^{-1} = \begin{bmatrix} \frac{1}{E} & 0 \\ 0 & \frac{1}{G} \end{bmatrix}$

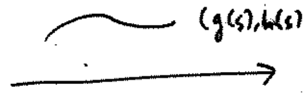
so $g^{11} = \frac{1}{E}$
$g^{12} = 0$

So $\Gamma_{11}^{1} = \frac{1}{2} \frac{1}{E} \left[g_{11,1} + g_{11,1} - g_{11,1} \right] = \frac{1}{2E} E_u$ ✓

$\Gamma_{11}^{2} = \frac{1}{2} g^{2\ell} \left(g_{\ell 1,1} + g_{\ell 1,1} - g_{11,\ell} \right)$

only $\ell = 2$ terms contribute

$= \frac{1}{2} \frac{1}{G} \left(0 + 0 - E_{,2} \right) = -\frac{E_v}{2G}$



3.3.13 $K = -\frac{h''}{h}$

for $K=+1$ $\quad h'' + h = 0$

$h = A\cos s + B \sin s = C\cos(s-s_0)$ ← for $C>0$ this is defined until $\cos(s-s_0)=0$ & curve crosses axis

$h' = -C\sin(s-s_0)$

$g'^2 + h'^2 = 1 \Rightarrow g' = \pm\sqrt{1-C^2\sin^2(s-s_0)} \Rightarrow g(s) = g_0 + \pm\int_0^s \sqrt{1-C^2\sin^2(r-s_0)}\,dr$

until $\sin^2(s-s_0) = \frac{1}{C^2}$

get sphere when e.g. $C=1 \Rightarrow$ ~~g(s) = ...~~

$h(s) = \cos(s-s_0)$, $g(s) = g_0 \pm \sin(s-s_0)$

3.4.5) $-\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right) \overset{?}{=} \frac{E_uG_u}{4E^2G} - \frac{1}{E}\left(\frac{E_v}{2G}\right)_v - \frac{E_vG_v}{4EG^2} + \frac{E_v^2}{4E^2G} - \frac{1}{E}\left(\frac{G_u}{2G}\right)_u - \frac{G_u^2}{4EG^2}$

$-\frac{1}{E}\left(\frac{E_{vv}}{2G} - \frac{E_vG_v}{2G^2}\right)$ like $\quad -\frac{1}{E}\left(\frac{G_{uu}}{2G} - \frac{G_u^2}{2G^2}\right)$ like

$$-\frac{1}{2\sqrt{EG}}\left[\frac{E_{vv}}{\sqrt{EG}} + \frac{E_v(-\frac{1}{2})(E_vG + EG_v)}{(EG)^{3/2}} + \frac{G_{uu}}{\sqrt{EG}} + \frac{G_u(-\frac{1}{2})(E_uG+EG_u)}{(EG)^{3/2}}\right]$$

$$\frac{E_uG_u}{4EG} - \frac{E_{vv}}{2EG} + \cancel{\frac{E_vG_v}{4EG^2}} + \cancel{\frac{E_v^2}{4E^2G}} - \cancel{\frac{1}{E}\frac{G_{uu}}{2G}} + \boxed{\frac{1}{4}\frac{G_u^2}{EG^2}}$$

$$-\frac{E_{vv}}{2EG} + \cancel{\frac{1}{4}\frac{E_v^2G}{(EG)^2}} + \cancel{\frac{1}{4}\frac{EE_vG_v}{E_G^2}} \cancel{-\frac{G_{uu}}{2EG}} + \frac{1}{4}\frac{E_uG_uG}{(EG)^2} + \boxed{\frac{1}{4}\frac{EG_u^2}{(EG)^2}}$$

3.4.6) Using Spherical coords

$X(u,v) = R\langle \sin v\cos u, \sin v \sin u, \cos v\rangle$

$X_u = R\langle -\sin v \sin u, \sin v\cos u, 0\rangle$

$X_v = R\langle \cos v\cos u, \cos v \sin u, -\sin v\rangle$

$X_u \cdot X_v = 0$

$X_u\cdot X_u = R^2\sin^2 v = E$

$X_v \cdot X_v = R^2 = G$

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$$

$$= -\frac{1}{2R^2\sin v}\left(\left(\frac{2R^2\sin v\cos v}{R^2 \sin v}\right)_v + 0\right)$$

$$= -\frac{1}{2R^2\sin v}\left(2(\cos v)_v\right) = \frac{--2\sin v}{2R^2\sin v} = \frac{1}{R^2}$$