Math 4530
HW set 5 ; due 2/28

Chptr 2: 2.15, 3.4, 3.9, 3.10, 3.11, 4.4, 4.6, 4.7
Chptr 3: 1.5, 1.6, 1.10, 1.11
2.4, 2.6 2.26, 2.13, 2.14, 2.15
3.3, 3.8

} exercises to hand in

2.2.15) Torus $X(u,v) = \langle (R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u \rangle$

$$X_u = \langle -r\sin u\cos v, -r\sin u\sin v, r\cos u \rangle$$
$$X_v = \langle (R + r\cos u)(-\sin v), (R + r\cos u)\cos v, 0 \rangle$$

$$\vec{U} \parallel \frac{1}{r}\vec{X}_u \times \frac{1}{R + r\cos u}\vec{X}_v : \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin u\cos v & -\sin u\sin v & \cos u \\ -\sin v & \cos v & 0 \end{vmatrix} = \langle -\cos v\cos u, -\sin v\cos u, -\sin u \rangle$$

is already a unit vector! $= \vec{U}$

$$\vec{U}_u = \langle \cos v\sin u, \sin v\sin u, -\cos u \rangle = -\frac{1}{r}X_u$$
$$\vec{U}_v = \langle \sin v\cos u, -\cos v\cos u, 0 \rangle = -\frac{\cos u}{R + r\cos u}X_v$$

$$\Rightarrow S(X_u) = -U_u = \frac{1}{r}X_u$$
$$S(X_v) = -U_v = \frac{\cos u}{R + r\cos u}X_v$$

3.4 a) $\{\vec{f}_1, \vec{f}_2\}$ o.n. basis.

$$[T]_{\{\vec{f}_1, \vec{f}_2\}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow T\vec{f}_1 = a\vec{f}_1 + c\vec{f}_2$$
$$T\vec{f}_2 = b\vec{f}_1 + d\vec{f}_2$$

T self adjoint means in particular
$$T\vec{f}_1 \cdot \vec{f}_2 = \vec{f}_1 \cdot T\vec{f}_2$$
$$c = d$$
(we did this in general in class – $n \times n$ case)

b) $\begin{vmatrix} a - \lambda & b \\ b & d - \lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + ad - b^2$

roots $\lambda = \dfrac{a + d \pm \sqrt{(a+d)^2 - 4(ad - b^2)}}{2}$

discriminant $\underbrace{(a+d)^2 - 4ad}_{} + 4b^2$
$= (a-d)^2 + 4b^2 \geq 0$, so roots are real

[one can then check that the (coordinates of) the corresponding eigenbasis vectors are $\perp$, in case $(a-d)^2 + 4b^2 > 0$

If $(a-d)^2 + 4b^2 = 0$ then $a = d$, $b = 0$
so matrix is multiple of identity and an o.n. basis is o.n. eigenbasis.]

2.3.9. $X(u,v) = \langle v\cos u, v\sin u, v\rangle$

$X_u = \langle -v\sin, v\cos, 0\rangle$

$X_v = \langle \cos u, \sin, 1\rangle$

$$\vec{U} \,/\!/\, \tfrac{1}{v} X_u \times X_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin & \cos & 0 \\ \cos & \sin & 1 \end{vmatrix} = \langle \cos u, \sin u, -1\rangle$$

$$\vec{U} = \tfrac{1}{\sqrt{2}} \langle \cos u, \sin u, -1\rangle$$

$x^2+y^2-z^2=0$

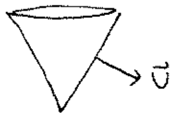


Image of Gauss map is equator at latitude 45° South ($\phi = 3\pi/4$ is spherical coords).

$\rightarrow$ $\boxed{\text{area of image is zero.}}$

$G_* = -S$

so $G_*(X_u) = U_u = \langle -\sin u, \cos u, 0\rangle = \frac{1}{v} X_u$

$G_*(X_v) = U_v = \langle 0,0,0\rangle$

$$[G_*]_{\{X_u, X_v\}} = \begin{bmatrix} 1/v & 0 \\ 0 & 0 \end{bmatrix}$$

det = 0

$\left(\iint_{\text{cone}} K\,dA = 0 = \pm \text{area of Gauss map image} \right).$

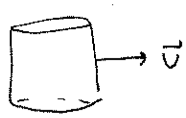
2.3.10. cylinder

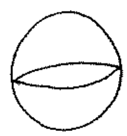
$X(u,v) = \langle R\cos u, R\sin u, v\rangle$

$X_u = \langle -R\sin, R\cos, 0\rangle$

$X_v = \langle 0, 0, 1\rangle$

$$\vec{U} \,/\!/\, \tfrac{1}{R} X_u \times X_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin & \cos & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos, \sin, 0\rangle = \vec{U}$$





$\leftarrow$ image of $\vec{U}$ is equator, as zero area.

$U_u = \langle -\sin u, \cos u, 0\rangle = \frac{1}{R} X_u$

$U_v = \langle 0,0,0\rangle$

$$[G_*]_{\{X_u, X_v\}} = \begin{bmatrix} 1/R & 0 \\ 0 & 0 \end{bmatrix}.$$

$\left(\iint_{\text{cyl}} K\,dA = 0 = \pm \text{ area of Gauss map image} \right)$

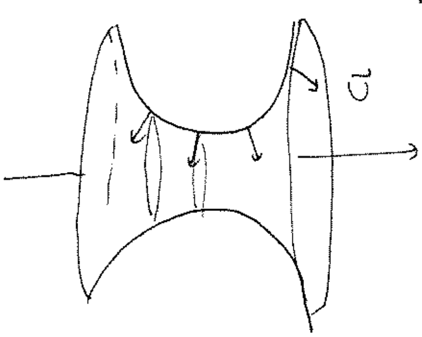
2.3.11 Catenoid $X(u,v) = \langle u, \cosh u \cos v, \cosh u \sin v\rangle$

$X_u = \langle 1, \sinh\cos, \sinh\sin\rangle$

$X_v = \langle 0, -\cosh\sin, \cosh\cos\rangle$

$$\vec{U} \,/\!/\, \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & \sinh\cos & \sinh\sin \\ 0 & -\cosh\sin & \cosh\cos \end{vmatrix} = \langle \sinh u\cosh u, -\cosh u\cos v, -\cosh u \sin v\rangle$$

$$/\!/ \ \langle \sinh u, -\cos v, -\sin v\rangle$$



"inner" normal $\vec{U} = \frac{1}{\sqrt{1+\sinh^2 u}} \langle \sinh u, -\cos v, -\sin v\rangle$

$= \frac{1}{\cosh u} \langle \sinh u, -\cos v, -\sin v\rangle$

image of normal map is all of S^2, covered exactly once.

$$U_u = \left\langle \frac{1}{\cosh^2 u}, \frac{+\cos v \sinh u}{\cosh^2 u}, \frac{\sin v \sinh u}{\cosh^2 u} \right\rangle = \frac{1}{\cosh^2 u} X_u$$

$$U_v = \frac{1}{\cosh u} \langle 0, \sin v, -\cos v\rangle = -\frac{1}{\cosh^2 u} X_v$$

[redacted]

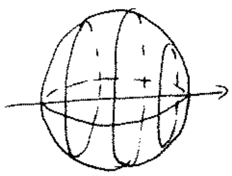
So $[G_*]_{\{X_u, X_v\}} = \begin{bmatrix} \frac{1}{\cosh^2 u} & 0 \\ 0 & -\frac{1}{\cosh^2 u} \end{bmatrix}$

$\vec{U}(u,v) = \left\langle \frac{\sinh u}{\cosh u}, \frac{-\cos v}{\cosh u}, \frac{-\sin v}{\cosh u} \right\rangle$

$\frac{d}{du} \frac{\sinh u}{\cosh u} = \frac{1}{\cosh^2 u} > 0$ so as $-\infty < u < \infty$, $-1 < \frac{\sinh u}{\cosh u} < 1$

increases monotonically,

so the normal traverses from the "west pole" to the "east pole", i.e. x coord increases from -1 to 1.



then the v coord determines uniquely where on the corresponding latitude $\vec{U}$ is, so $\vec{U}$ is 1-1.

$\left(\vec{U}(u_1,v_1) = \vec{U}(u_2,v_2) \Rightarrow \tanh u_1 = \tanh u_2 \Rightarrow u_1 = u_2 \right.$
$\left. \Rightarrow \begin{matrix} \cos v_1 = \cos v_2 \\ \sin v_1 = \sin v_2 \end{matrix} \right\} \Rightarrow v_1 = v_2 \Big).$

2.4.4 done in class!
(Fri 2/25 notes)

2.4.6 Method A): $\vec{u} = \langle 0,1,0 \rangle = \vec{e}_2$

$k(\vec{u}) = S(\vec{u}) \cdot \vec{u}$

$[S]_{\{\vec{e}_1, \vec{e}_2\}} = \begin{bmatrix} f_{xx}(\vec{0}) & f_{xy}(\vec{0}) \\ f_{yx}(\vec{0}) & f_{yy}(\vec{0}) \end{bmatrix}$ Because $f_x(\vec{0}) = f_y(\vec{0}) = 0$ (graphing above tang. plane)

$f(x,y) = x^2 - y^2$

$= \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

$f_{xx} = 2$ $f_{yy} = -2$
$f_{xy} = 0$

so $S(\vec{e}_2) = -2\vec{e}_2$
$S(\vec{e}_2) \cdot \vec{e}_2 = -2$

Method C) $k_n(\vec{u})$ is $\pm$ the curvature of the curve ~~through~~ of intersection btwn M and the $\vec{U}, \vec{v}$ plane. This curve is $z = -y^2 = g(y)$. curvature at $y=0$ is $g''(0) = -2$, relative to $\vec{U}$

Method B) $S(\vec{u}) = \alpha''(0) \cdot \vec{U}$

where α is any curve: $\alpha(0) = (0,0,0)$
$\alpha'(0) = (0,1,0)$
α lies on M^2.

e.g. $\alpha(y) = (0, y, -y^2)$
$\alpha'(y) = (0, 1, -2y)$
$\alpha''(y) = (0, 0, -2)$

$\langle 0,0,-2 \rangle \cdot \langle 0,0,1 \rangle = -2$

2.4.7 Method A) $\vec{u} = \frac{1}{\sqrt{2}}\vec{e}_1 + \frac{1}{\sqrt{2}}\vec{e}_2$

$S(\vec{u}) \cdot \vec{u} = \Big(\frac{1}{\sqrt{2}} \underbrace{S(\vec{e}_1)}_{2\vec{e}_1} + \frac{1}{\sqrt{2}} \underbrace{S(\vec{e}_2)}_{-2\vec{e}_2} \Big) \cdot \Big(\frac{1}{\sqrt{2}}\vec{e}_1 + \frac{1}{\sqrt{2}}\vec{e}_2 \Big)$

from Hessian.

$= 1 - 1 = 0$

Method C) the vertical plane $y = x$ has trace curve $z = 0$ with M. This curve is straight so $k_n(\vec{u}) = 0$

$\left(\frac{1}{\sqrt{2}}t\right)^2 - \left(\frac{1}{\sqrt{2}}t\right)^2 = 0$

Method B) $\alpha(t) = \langle \frac{1}{\sqrt{2}}t, \frac{1}{\sqrt{2}}t, 0 \rangle$

$\alpha'(t) = \langle 1/\sqrt{2}, 1/\sqrt{2}, 0 \rangle$

$\alpha''(t) = \vec{0}$, $\vec{\alpha}''(0) \cdot \vec{U} = 0$

Chptr 3

1.5 : Maple could do this in a flash!

```
> dp:=proc(X,Y)  #dotproduct
  X[1]*Y[1]+X[2]*Y[2]+X[3]*Y[3];
  end:
  xp:=proc(X,Y)  #cross product
  local a,b,c;
  a:=X[2]*Y[3]-X[3]*Y[2];
  b:=X[3]*Y[1]-Y[3]*X[1];
  c:=X[1]*Y[2]-X[2]*Y[1];
  [a,b,c];
  end:
> v:=[v1,v2,v3]:
  w:=[w1,w2,w3]:
  a:=[a1,a2,a3]:
  b:=[b1,b2,b3]:
> simplify(dp(xp(v,w),xp(a,b))-
  dp(v,a)*dp(w,b)+dp(v,b)*dp(w,a));
  #If we get zero, Lagrange holds!
                                    0
>
```

But so can we.

write $\vec{v} = \sum v^i \vec{e}_i$

$\vec{w} = \sum w^j \vec{e}_j$

$\vec{a} = \sum a^k \vec{e}_k$

$\vec{b} = \sum b^l \vec{e}_l$.

By multilinearity of

$LHS(\vec{v},\vec{w},\vec{a},\vec{b}) = (\vec{v}\times\vec{w})\cdot(\vec{a}\times\vec{b})$

$RHS(\vec{v},\vec{w},\vec{a},\vec{b}) = \begin{vmatrix} \vec{v}\cdot\vec{a} & \vec{v}\cdot\vec{b} \\ \vec{w}\cdot\vec{a} & \vec{w}\cdot\vec{b}\end{vmatrix}$

Deduce

$$LHS(\vec{v},\vec{w},\vec{a},\vec{b}) = \sum_{i,j,k,l} v^i w^j a^k b^l \, (\vec{e}_i\times\vec{e}_j)\cdot(\vec{e}_k\times\vec{e}_l)$$

$$RHS(\vec{v},\vec{w},\vec{a},\vec{b}) = \sum v^i w^j a^k b^l \begin{vmatrix} \vec{e}_i\cdot\vec{e}_k & \vec{e}_i\cdot\vec{e}_l \\ \vec{e}_j\cdot\vec{e}_k & \vec{e}_j\cdot\vec{e}_l\end{vmatrix}$$

so suffices to show

$$(\vec{e}_i\times\vec{e}_j)\cdot(\vec{e}_k\times\vec{e}_l) = \begin{vmatrix} \vec{e}_i\cdot\vec{e}_k & \vec{e}_i\cdot\vec{e}_l \\ \vec{e}_j\cdot\vec{e}_k & \vec{e}_j\cdot\vec{e}_l\end{vmatrix}$$

both sides are antisymmetric in (i,j) and in (k,l).
so will $=0$ if $i=j$ or $k=l$,
and so we can assume
$i<j$, $k<l$

Since $\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ are orthonormal,

LHS $\neq 0$ only if $(i,j)=(k,l)$
and then its value is $(\vec{e}_i\times\vec{e}_j)\cdot(\vec{e}_i\times\vec{e}_j) = 1$.

RHS $\neq 0$ only if $(i,j)=(k,l)$, since otherwise det matrix will have 3 zero's & one 1, $\Rightarrow \det = 0$

if $(i,j)=(k,l)$

$\det = \begin{vmatrix} \vec{e}_i\cdot\vec{e}_i & \vec{e}_i\cdot\vec{e}_j \\ \vec{e}_j\cdot\vec{e}_i & \vec{e}_j\cdot\vec{e}_j\end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1\end{vmatrix} = 1$ $\blacksquare$

1.6. $S\{\vec{f}_1\} = k_1\vec{f}_1$

$S(\vec{f}_2\} = k_2\vec{f}_2$

$\vec{f}_1, \vec{f}_2$ o.n.

then if $k(\theta) := k_n(\cos\theta\vec{f}_1 + \sin\theta\vec{f}_2)$

$= S(\cos\theta\vec{f}_1 + \sin\theta\vec{f}_2)\cdot(\cos\theta\vec{f}_1 + \sin\theta\vec{f}_2) = (\cos\theta\, k_1\vec{f}_1 + \sin\theta\, k_2\vec{f}_2)\cdot(\cos\theta\vec{f}_1 + \sin\theta\vec{f}_2)$

$= \cos^2\theta\, k_1 + \sin^2\theta\, k_2$ (Euler's formula)

So

(a) $\frac{1}{2\pi}\int_0^{2\pi} k(\theta)\,d\theta = \frac{1}{2\pi}\left[\left(\int_0^{2\pi}\cos^2\theta\,d\theta\right)k_1 + \left(\int_0^{2\pi}\sin^2\theta\,d\theta\right)k_2\right] = \frac{1}{2\pi}(\pi k_1 + \pi k_2) = \frac{1}{2}(k_1+k_2) = H$

(b) $k(\theta) + k(\theta+\pi/2) = \cos^2\theta\, k_1 + \sin^2\theta\, k_2 + \cos^2(\theta+\pi/2)\, k_1 + \sin^2(\theta+\pi/2)\, k_2$

but $\cos(\theta+\pi/2) = -\sin\theta$, $\sin(\theta+\pi/2) = \cos\theta$, so get $k_1(\cos^2\theta+\sin^2\theta) + k_2(\sin^2\theta+\cos^2\theta) = k_1+k_2$

so $\frac{1}{2}(k(\theta)+k(\theta+\pi/2)) = H$ $\blacksquare$

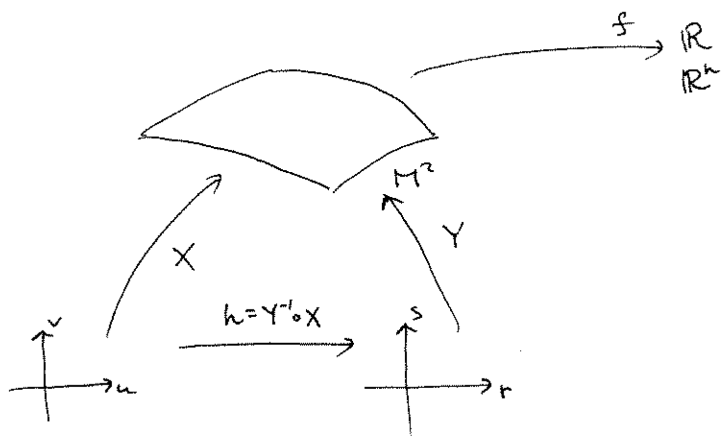
3.1.10 M^2 minimal means $H=0 \Rightarrow k_1+k_2=0$
$$\Rightarrow k_2=-k_1$$
$$\Rightarrow K=-k_1^2 \le 0$$

3.1.11 Since we may take
$U=\frac{1}{R}X(u,v)$ for any chart on the sphere

$$\rightarrow \begin{array}{l} S(X_u)=-U_u=-\frac{1}{R}X_u \\ S(X_v)=-U_v=-\frac{1}{R}X_v \end{array} \Rightarrow [S]_{\{X_u,X_v\}}=\begin{bmatrix}-1/R & 0\\ 0 & -1/R\end{bmatrix} \Rightarrow K=\frac{1}{R^2}$$

alternately U shrinks the R-sphere to the unit sphere, so shrinks area by $\frac{1}{R^2}$
i.e. $dA_{S^2}=\frac{1}{R^2}dA_M \Rightarrow K=\frac{1}{R^2}$.

2.4 This exercise can be done as part of a larger chain rule discussion



$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial r}\frac{\partial r}{\partial u}+\frac{\partial f}{\partial s}\frac{\partial s}{\partial u} \quad ; \quad \frac{\partial f}{\partial v}=\frac{\partial f}{\partial r}\frac{\partial r}{\partial v}+\frac{\partial f}{\partial s}\frac{\partial s}{\partial v}$$

Special cases

$X=Y\circ h$: $\vec{X}_u=\vec{Y}_r r_u+\vec{Y}_s s_u$; $\vec{X}_v=\vec{Y}_r r_v+\vec{Y}_s s_v$

$$\left[\vec{X}_u \middle| \vec{X}_v\right]=\left[\vec{Y}_r \middle| \vec{Y}_s\right]\begin{bmatrix} r_u & r_v\\ s_u & s_v\end{bmatrix}$$ write $[X'(u,v)]=[Y'(r,s)]\,[h'(r,s)]$

$U=U\circ h$ $$\left[\vec{U}_u \middle| \vec{U}_v\right]=\left[\vec{U}_r \middle| \vec{U}_s\right]\begin{bmatrix} r_u & r_v\\ s_u & s_v\end{bmatrix}$$ write $[U'(u,v)]=[U'(r,s)][h'(r,s)]$

Thus $[I]_{\{\vec{X}_u,\vec{X}_v\}}=[X']^T[X']=[h']^T[Y']^T[Y'][h]=[h']^T[I]_{\{Y_r,Y_s\}}[h']$

And $[II]_{\{\vec{X}_u,\vec{X}_v\}}=[X']^T[U'_{(u,v)}]=[h']^T[Y']^T[U'(r,s)][h']=[h']^T[II]_{\{Y_r,Y_s\}}[h']$

Finally, $K=\dfrac{\det[II]_{\{X_u,X_v\}}}{\det[I]_{\{X_u,X_v\}}}=\dfrac{\cancel{(\det[h'])^2}\det[II]_{\{\vec{Y}_r,\vec{Y}_s\}}}{\cancel{(\det[h'])^2}\det[I]_{\{\vec{Y}_r,\vec{Y}_s\}}}$

2.6 $X^t(u,v) = X(u,v) + t\,U(u,v)$

$$\begin{aligned} X^t_u &= X_u + t\,U_u \\ X^t_v &= X_v + t\,U_v \end{aligned} \quad \Rightarrow X^t_u \times X^t_v \,/\!/\, X_u \times X_v \quad \text{because } X^t_u, X^t_v \in T_pM.$$

$\Rightarrow U^t = U$ as long as X^t remains regular.

Notice, the parallel surface construction, and the desired result, is a property of the surfaces, - not the particular patch.

So, near p, assume $X(u,v) = u\vec{f}_1 + v\vec{f}_2 + g(u,v)\vec{f}_3 + p$, is a parameterization above T_pM $(\vec{f}_3 = \vec{f}_1 \times \vec{f}_2)$,

s.t. $\{\vec{f}_1, \vec{f}_2\}$ is o.n. eigenbasis of S_p

so $g(0,0)=0$
$g_u(0,0) = g_v(0,0) = 0$.

at P, when $t=0$, So
$$\begin{cases} [\mathrm{I}] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ [\mathrm{II}] = \begin{bmatrix} g_{uu} & g_{uv} \\ g_{vu} & g_{vv} \end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} = [S]_{\{\vec{f}_1, \vec{f}_2\}} \end{cases}$$

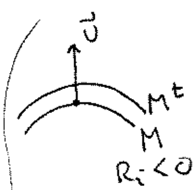
minus sign because $U_u = -S(X_u) = -k_1 X_u$

Now, for $(u,v)=(0,0)$
$$X^t_u = X_u + t\,U_u = \vec{f}_1 - t k_1 \vec{f}_1 = \vec{f}_1(1 - t k_1)$$
$$X^t_v = X_v + t\,U_v = \vec{f}_2(1 - t k_2)$$
$$\Rightarrow [\mathrm{I}^t] = \begin{bmatrix} (1 - t k_1)^2 & 0 \\ 0 & (1 - t k_2)^2 \end{bmatrix}$$

$U^t = U$

so $U^t_u = U_u$; $S^t(X^t_u) = -U_u = k_1 \vec{f}_1$
$U^t_v = U_v$ $\quad S^t(X^t_v) = -U_v = k_2 \vec{f}_2$

$$\Rightarrow [\mathrm{II}^t] = \begin{bmatrix} k_1(1 - t k_1) & 0 \\ 0 & k_2(1 - t k_2) \end{bmatrix}$$



$$\Rightarrow [S^t] = [\mathrm{I}^t]^{-1}[\mathrm{II}^t]$$
$$= \frac{1}{(1-tk_1)^2(1-tk_2)^2} \begin{bmatrix} (1-tk_2)^2 & 0 \\ 0 & (1-tk_1)^2 \end{bmatrix} \begin{bmatrix} k_1(1+tk_1) & 0 \\ 0 & k_2(1+tk_2) \end{bmatrix}$$

$$[S^t]_{\{X^t_u, X^t_v\}} = \begin{bmatrix} \dfrac{k_1}{1-tk_1} & 0 \\ 0 & \dfrac{k_2}{1-tk_2} \end{bmatrix}$$

$\leftarrow$ principal directions are in X^t_u, X^t_v dirs!

radii of curvature

$R^t_i = \dfrac{1-tk_i}{k_i} = R_i - t$: think of circles!

(also R_i could be negative)

Thus $K^t = \dfrac{k_1 k_2}{(1-tk_1)(1-tk_2)} = \dfrac{K}{1-2tH+t^2K}$

$$H^t = \frac{1}{2}\left(\frac{k_1}{1-tk_1} + \frac{k_2}{1-tk_2}\right) = \frac{1}{2}\,\frac{k_1(1-tk_2)+k_2(1-tk_1)}{(1-tk_1)(1-tk_2)}$$

$$= \frac{\frac{1}{2}\left[k_1+k_2-2tK\right]}{1-2tH+t^2K} = \frac{H-Kt}{1-2tH+t^2K} \qquad \blacksquare$$

2.26. $M: z = f(x,y)$

a) $X(u,v) = \langle u, v, f(u,v)\rangle$

b) $X_u = \langle 1, 0, f_u\rangle$

$X_v = \langle 0, 1, f_v\rangle$

$$U = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{\langle -f_u, -f_v, 1\rangle}{\sqrt{1+f_u^2+f_v^2}}$$

$$[I]_{\{X_u, X_v\}} = \begin{bmatrix} 1+f_u^2 & f_u f_v \\ f_v f_u & 1+f_v^2 \end{bmatrix}$$

$X_{uu} = \langle 0, 0, f_{uu}\rangle$

$X_{uv} = \langle 0, 0, f_{uv}\rangle$

$X_{vv} = \langle 0, 0, f_{vv}\rangle$

$$[II]_{\{X_u, X_v\}} = \frac{1}{\sqrt{1+f_u^2+f_v^2}}\begin{bmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{bmatrix}$$

$$[S] = [I]^{-1}[II]$$

note, $\det[I] = (1+f_u^2)(1+f_v^2) - f_u^2 f_v^2$

$= 1 + f_u^2 + f_v^2$

$$[S] = \frac{1}{1+f_u^2+f_v^2}\begin{bmatrix} 1+f_v^2 & -f_u f_v \\ -f_v f_u & 1+f_u^2 \end{bmatrix} \frac{1}{\sqrt{1+f_u^2+f_v^2}}\begin{bmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{bmatrix}$$

$$= \frac{1}{(1+f_u^2+f_v^2)^{3/2}}\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

c) $K = \det[S]$

$$= \frac{\det[II]}{\det[I]}$$

$$= \frac{\frac{1}{1+f_u^2+f_v^2}\left(f_{uu}f_{vv} - f_{uv}^2\right)}{1+f_u^2+f_v^2}$$

$$K = \frac{f_{uu}f_{vv} - f_{uv}^2}{(1+f_u^2+f_v^2)^2}$$

$$H = \frac{\operatorname{trace} S}{2} = \frac{1/2}{(1+f_u^2+f_v^2)^{3/2}}\left[f_{uu}(1+f_v^2) - 2f_u f_v f_{uv} + f_{vv}(1+f_u^2)\right] = \frac{f_{uu}(1+f_v^2) - 2f_u f_v f_{uv} + f_{vv}(1+f_u^2)}{2(1+f_u^2+f_v^2)^{3/2}}$$

(d) $D > 0$ iff $K > 0$. At critical pt $K = f_{uu}f_{vv} - f_{uv}^2 = D$

$H = \frac{1}{2}(f_{uu} + f_{vv})$

(k_i = roots of $\lambda^2 - 2H\lambda + K = 0$
$\lambda = H \pm \sqrt{H^2 - K}$

(I) $K > 0$, $f_{uu} > 0 \Rightarrow f_{vv} > 0$ (see K) $\Rightarrow H > 0$; $K, H > 0 \Rightarrow k_1, k_2 > 0$

(II) $K > 0$, $f_{uu} < 0 \Rightarrow f_{vv} < 0 \Rightarrow H < 0 \Rightarrow k_1, k_2 < 0$.

2.13 $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

apply 2.26 $$K = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2} = \frac{\frac{2}{a^2}\frac{2}{b^2}}{\left(1 + \frac{4x^2}{a^4} + \frac{4y^2}{b^4}\right)^2} = \frac{1}{\frac{a^2b^2}{4}\left(1 + \frac{4x^2}{a^4} + \frac{4y^2}{b^4}\right)^2}$$

$$= \frac{1}{4a^2b^2\left(\frac{1}{4} + \frac{x^2}{a^4} + \frac{y^2}{b^4}\right)^2}$$

2.14 $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

apply 2.26 $$K = \frac{-\frac{2}{a^2}\frac{2}{b^2}}{\left(1 + \frac{4x^2}{a^4} + \frac{4y^2}{b^4}\right)^2}$$

$$= -\frac{1}{4a^2b^2\left(\frac{1}{4} + \frac{x^2}{a^4} + \frac{y^2}{b^4}\right)^2}$$

2.15 $X(u,v) = \langle u\cos v, u\sin v, bv\rangle$

$X_u = \langle \cos, \sin, 0\rangle$

$X_v = \langle -u\sin, u\cos, b\rangle$

$$\vec{U} \parallel \vec{X}_u \times \vec{X}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos & \sin & 0 \\ -u\sin & u\cos & b \end{vmatrix} = \langle +b\sin, -b\cos, u\rangle$$

$$\vec{U} = \frac{1}{\sqrt{u^2+b^2}}\langle +b\sin v, -b\cos v, u\rangle$$

$X_{uu} = 0$

$X_{uv} = \langle -\sin v, \cos v, 0\rangle$

$X_{vv} = \langle -u\cos v, -u\sin v, 0\rangle$

$$[I] = \begin{bmatrix} 1 & 0 \\ 0 & b^2+u^2 \end{bmatrix}$$

$$[II] = \begin{bmatrix} 0 & \frac{-b}{\sqrt{u^2+b^2}} \\ \frac{-b}{\sqrt{u^2+b^2}} & 0 \end{bmatrix}$$

$$[S]_{\{X_u, X_v\}} = [I]^{-1}[II] = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{b^2+u^2} \end{bmatrix} \begin{bmatrix} 0 & -\frac{b}{\sqrt{\ }} \\ \frac{-b}{\sqrt{\ }} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -b/\sqrt{u^2+b^2} \\ \frac{-b}{(b^2+u^2)^{3/2}} & 0 \end{bmatrix}$$

$$\boxed{\Rightarrow K = \frac{-b^2}{(b^2+u^2)^2} \quad H = 0}$$

3.3 Using 2.2.15, $[S]_{\{X_u,X_v\}} = \begin{bmatrix} -\frac{1}{r} & 0 \\ 0 & -\frac{\cos u}{R+r\cos u} \end{bmatrix}$

$$\Rightarrow K = \frac{\cos u}{r(R+r\cos u)} \;; \quad \text{So } K\Big|_{u=0} = \frac{1}{r(R+r)}$$

$$K\Big|_{u=\pi} = -\frac{1}{r(R-r)}$$

3.8. From class notes Fri 2/18.

$X(u,v) = \langle x(u), y(u)\cos v, y(u)\sin v\rangle = \langle g(u), h(u)\cos v, h(u)\sin v\rangle$

$$[I]_{\{X_u,X_v\}} = \begin{bmatrix} (x')^2+(y')^2 & 0 \\ 0 & y^2 \end{bmatrix}$$

so if $\langle x(u), y(u)\rangle$ pbal, $[I] = \begin{bmatrix} 1 & 0 \\ 0 & y^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & h^2 \end{bmatrix}$

& $[II]_{\{X_u,X_v\}} = \frac{1}{\sqrt{g'^2+h'^2}} \begin{bmatrix} g''h'-g'h'' & 0 \\ 0 & hg' \end{bmatrix}$

if $\langle g,h\rangle$ pbal

$$[II] = \begin{bmatrix} g''h'-g'h'' & 0 \\ 0 & hg' \end{bmatrix}$$

so $[S] = [I]^{-1}[II]$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1/h^2 \end{bmatrix} \begin{bmatrix} g''h'-g'h'' & 0 \\ 0 & hg' \end{bmatrix}$$

$$\Rightarrow K = \frac{1}{h^2}(g''h'-g'h'')hg'$$

$$= \frac{g'}{h}(g''h'-g'h'').$$

But $(g')^2+(h')^2 \equiv 1$

$\Rightarrow g'g''+h'h'' = 0$

so $K = \frac{1}{h}\left[(-h'h'')h' - \underbrace{(g')^2}_{1-h'^2}h''\right]$

$$= \frac{1}{h}\left[h''(-h'^2-1+h'^2)\right]$$

$$= \frac{-h''}{h} \quad !$$