Math 4530
HW set 4 - due 2/11
See also Maple file.

2.1.22 $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

We try for a ruling from the base curve
$$\beta(u) = \langle a\cos u, b\sin u, 0\rangle,$$
the waist of the hyperboloid.
We expect the line direction to have horizontal component // to $\beta'(u)$.
In fact, book says to take
$$\delta(u) = \beta'(u) + \langle 0,0,c\rangle.$$

Let's try it:
$$X(u,v) = \beta(u) + v\,\delta(u)$$
$$= \langle a\cos u, b\sin u, 0\rangle + v\langle -a\sin u, b\cos u, c\rangle$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \frac{\cancel{a^2}(\cos u - v\sin u)^2}{\cancel{a^2}} + \frac{\cancel{b^2}(\sin u + v\cos u)^2}{\cancel{b^2}} - \frac{\cancel{c^2}v^2}{\cancel{c^2}}$$
$$= \cos^2 u - \cancel{2v\cos u\sin u} + \cancel{v^2\sin^2 u} + \sin^2 u + \cancel{2v\sin u\cos u} + \cancel{v^2\cos^2 u} - \cancel{v^2} = 1, \ !$$

- In fact this is a double ruling because we could take $\delta(u) = \beta'(u) + \langle 0,0,-c\rangle$ as well — the computation above would be virtually identical.

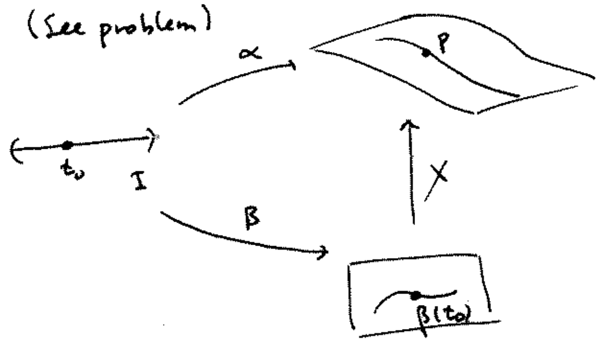
2.7 $X_u(f) = \frac{\partial}{\partial u} f\circ X$
$$= \sum_i \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial u}$$
so $X_v(X_u(f)) = X_v\left(\sum \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial u}\right) = \sum_i \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial x^i}\right)\frac{\partial x^i}{\partial u} + \sum_i \frac{\partial f}{\partial x^i}\frac{\partial^2 x^i}{\partial v\partial u}$
$$= \sum_{i,j} \frac{\partial^2 f}{\partial x^j\partial x^i}\frac{\partial x^j}{\partial v}\frac{\partial x^i}{\partial u} + \sum_i \frac{\partial f}{\partial x^i}\frac{\partial^2 x^i}{\partial v\partial u}$$

Notice $X_v(X_u(f)) = X_u(X_v(f))$,
and this must be true, since one equals $(f\circ X)_{uv}$, the other $(f\circ X)_{vu}$

2.9. $Z = \sum Z^k \vec{e}_k$
$$\nabla_{X_u} Z = \sum_k X_u(Z^k)\vec{e}_k = \sum_k (Z^k\circ X)_u \vec{e}_k = \sum_{k,i} \frac{\partial Z^k}{\partial x^i}\frac{\partial x^i}{\partial u}\vec{e}_k$$
so $\nabla_{X_v}(\nabla_{X_u}\vec{Z}) = \sum_{k,i} X_v\left(\frac{\partial Z^k}{\partial x^i}\frac{\partial x^i}{\partial u}\right)\vec{e}_k = \sum_{k,i,j}\left[\frac{\partial^2 Z^k}{\partial x^j\partial x^i}\frac{\partial x^j}{\partial v}\frac{\partial x^i}{\partial u}\right]\vec{e}_k + \sum_{k,i}\left[\frac{\partial Z^k}{\partial x^i}\frac{\partial^2 x^i}{\partial v\partial u}\right]\vec{e}_k$

Notice $\nabla_{\vec{X}_v}\nabla_{\vec{X}_u}\vec{Z} = \nabla_{\vec{X}_u}\nabla_{\vec{X}_v}\vec{Z}$, as it must since one equals $\vec{Z}_{uv}$, the other $\vec{Z}_{vu}$

I) (See problem)



$\beta := X^{-1}\circ\alpha$ (where it makes sense).

We must show that if α is diff'ble in Chapter 1 sense, as a map to $\mathbb{R}^3$ (which happens to lie on M), then $\beta := X^{-1}\circ\alpha$ is too.

proof Near $(\beta(t_0), 0)$ define $Z(u,v,w) = X(u,v) + w\left(\frac{X_u \times X_v}{\|X_u \times X_v\|}\right)$

$\overset{\shortparallel}{(q,0)}$ So $Z(u,v,0)$ is original $X(u,v)$

the derivative matrix of Z at $(u,v,0) = (q,0)$ is

$$\left[X_u \,\vdots\, X_v \,\vdots\, U\right] \quad \text{where } U = \frac{X_u \times X_v}{\|X_u \times X_v\|}, \text{ at } q.$$

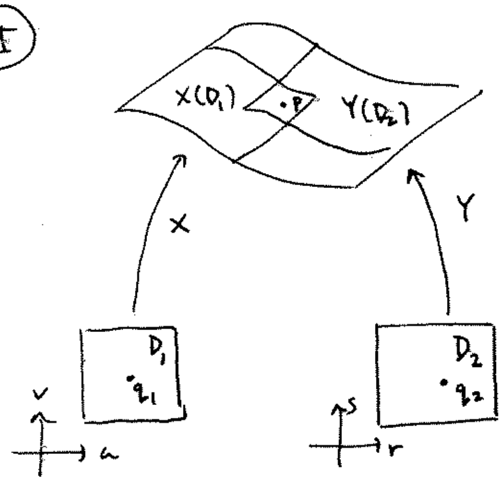
This matrix is nonsingular at $(q,0)$ because X_u, X_v, U are lin. ind.

($c_1 X_u + c_2 X_u + c_3 U = 0$

dot with $U \Rightarrow c_3 = 0$
then $c_1 = c_2 = 0$ since X_u, X_v l.i.)

Thus Inverse fcn thm implies
Z has a local inverse fcn (which is ∞'ly diff'ble since Z is),
mapping a neighborhood of $p = X(q)$ back to a neighborhood of $(q,0)$

But $X\circ\beta = \alpha$
$\Rightarrow$ $Z\circ\beta = \alpha$
$\Rightarrow$ $\beta = Z^{-1}\circ\alpha$ is composition of diff'ble is diff'ble ∎



show $Y^{-1}\circ X : X^{-1}(Y(D_2)) \to Y^{-1}(X(D_1))$

is diff'ble.

Same fattening trick: note, differentiability is just a local property.

The problem here is Y^{-1}, since X is diff'ble.

So let $p \in X(D_1) \cap Y(D_2)$

$$q_1 = X^{-1}(p)$$
$$q_2 = Y^{-1}(p)$$

Near $(q_2, 0)$ define

$$Z(r,s,w) = Y(r,s) + w\left(\frac{Y_r \times Y_s}{\|Y_r \times Y_s\|}\right)$$

Deriv matrix of Z at $(q_2, 0)$ is

$$\left[\, Y_r \,\middle|\, Y_s \,\middle|\, U(r,s) \,\right]$$

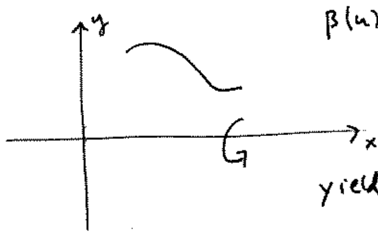
all entries evaluated at $(u,v) = q_2$.

Since matrix is nonsingular at $(q_2, 0)$, Z has local inverse, so near $(q_2, 0)$ and p, and q_1,

$$Y^{-1} \circ X = Z^{-1} \circ X$$

is a composition of diff'ble maps is diff'ble ■

Monge patches for surfaces of revolution



$\beta(u) = \langle f(u), g(u)\rangle$

Assume $g > 0$

β regular

β 1-1.

yields $X(u,v) = \langle f(u), g(u)\cos v, g(u)\sin v\rangle$ $0 < v < 2\pi$, e.g. (any open interval of length $\leq 2\pi$ would work)

X 1-1: $X(u,v) = X(\bar{u},\bar{v}) \Rightarrow f(u) = f(\bar{u})$ (x-cond)

$\Rightarrow g(u)\cos v = g(\bar{u})\cos\bar{v}$

$\quad g(u)\sin v = g(\bar{u})\sin\bar{v}$

$\Rightarrow g(u)^2 = g(\bar{u})^2$ (compare y^2+z^2 to $\bar{y}^2+\bar{z}^2$)

$\Rightarrow g(u) = g(\bar{u})$

$\Rightarrow u = \bar{u}$ (since $\langle f(u), g(u)\rangle = \langle f(\bar{u}), g(\bar{u})\rangle$)

$\Rightarrow v = \bar{v}$ mod 2π since $\cos v = \cos\bar{v}$, $\sin v = \sin\bar{v}$ ■

X is regular:

X is diff'ble.

$X_u = \langle f'(u), g'(u)\cos v, g'(u)\sin v\rangle$

$X_v = \langle 0, g(-\sin v), g\cos v\rangle$

$\langle f, g\rangle$ regular $\Rightarrow \langle f', g'\rangle \neq \vec{0}$

$\Rightarrow X_u \neq 0$

$g > 0 \Rightarrow X_v \neq 0$

$\vec{X}_u \cdot X_v = 0$ so X_u, X_v lin. ind. ■