HW set2 – Math 4530
Due 1/28, Chapter I problems

3.19, 3.22, 3.27, 3.28   4.4, 4.6, 4.7   5.3, 5.4, 5.6, 5.7   Class exercise I from jan24.pdf

3.19) $\beta(s) = \begin{bmatrix} a\cos s/c \\ a \sin s/c \\ \frac{bs}{c} \end{bmatrix}$ $c = \sqrt{a^2+b^2}$

as in book:

$\beta' = T = \begin{bmatrix} -\frac{a}{c}\sin\frac{s}{c} \\ \frac{a}{c}\cos\frac{s}{c} \\ \frac{b}{c} \end{bmatrix}$

$T' = \kappa N = \begin{bmatrix} -\frac{a}{c^2}\cos \\ -\frac{a}{c^2}\sin \\ 0 \end{bmatrix} = \underset{\kappa}{\frac{a}{c^2}} \underset{N}{\begin{bmatrix} -\cos\frac{s}{c} \\ -\sin\frac{s}{c} \\ 0 \end{bmatrix}}$

$B = T \times N = \begin{bmatrix} i & j & k \\ -\frac{a}{c}\sin & \frac{a}{c}\cos & \frac{b}{c} \\ -\cos\frac{s}{c} & -\sin\frac{s}{c} & 0 \end{bmatrix}$

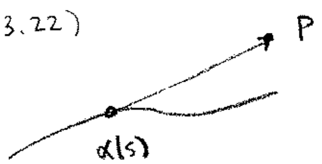
$\boxed{B = \begin{bmatrix} \frac{b}{c}\sin \\ -\frac{b}{c}\cos \\ \frac{a}{c} \end{bmatrix}}$

$\boxed{N' = -\kappa T + \tau B = \frac{1}{c}\begin{bmatrix} \sin\frac{s}{c} \\ -\cos\frac{s}{c} \\ 0 \end{bmatrix}}$

so $\tau = N' \cdot B$

$= \frac{1}{c}\left[\frac{b}{c}\sin^2 + \frac{b}{c}\cos^2\right] = \frac{b}{c^2} = \boxed{\frac{b}{a^2+b^2} = \tau}$

3.22)



let α pbal

$P = \alpha(s) + r(s)T(s)$ $T(s) = \alpha'(s)$

$r(s) = (P - \alpha(s)) \cdot T(s)$ is diffble

$\frac{d}{ds}$: $0 = \alpha' + r'T + rT'$

$\vec{0} = T + r'T + r(\kappa N)$

$* \quad 0 \equiv (1+r')T + r\kappa N$

$\Rightarrow 1 + r'(s) \equiv 0$ (dot * with T) $\Rightarrow \boxed{r(s) = -s + c}$

$r\kappa \equiv 0$ (" " " N) $\Rightarrow (-s+c)\kappa(s) \equiv 0$

$\boxed{\Rightarrow \kappa(s) \equiv 0}$ (if κ cont, i.e. curve C^∞)

$\Rightarrow P = \alpha(s) + (-s+c)T(s)$

but $T' = \kappa N \equiv 0$ so $T(s) \equiv T_0$ const

$\Rightarrow P = \alpha(s) + (-s+c)T_0$

$\boxed{\alpha(s) = P + (s-c)T_0}$; straight line in dir of T_0, thru P.

3.27: We saw this circle before; it's the intersection of the $x=z$ plane with the unit sphere, so is a circle of radius 1

$$\Rightarrow \kappa = 1/R = 1$$
$$\tau = 0 \quad \text{(planar)}.$$

If we have a bad memory check

$$\beta(s) = \begin{bmatrix} \frac{1}{\sqrt{2}}\cos s \\ \sin s \\ \frac{1}{\sqrt{2}}\cos s \end{bmatrix}, \quad \beta'(s) = \begin{bmatrix} -\frac{1}{\sqrt{2}}\sin s \\ \cos s \\ -\frac{1}{\sqrt{2}}\sin s \end{bmatrix} \qquad \|\beta'\| = 1 \text{ so } \beta \text{ pbal}$$

$$T' = \beta'' = \begin{bmatrix} -\frac{1}{\sqrt{2}}\cos s \\ -\sin s \\ -\frac{1}{\sqrt{2}}\cos s \end{bmatrix} \qquad \|\beta''\| = \kappa = 1 \checkmark$$

and $N = T'$

$$\Rightarrow N' = T'' = \beta''' = \begin{bmatrix} \frac{1}{\sqrt{2}}\sin s \\ -\cos s \\ \frac{1}{\sqrt{2}}\sin s \end{bmatrix} = -T$$

but $N' = -T$

frenet $\Rightarrow N' = -\kappa T + \tau B \Rightarrow \tau B = 0$
$\Rightarrow \tau = 0$

3.28:

circle: $\beta(s) = P + R\cos\frac{s}{R}\vec{v}_1 + R\sin\frac{s}{R}\vec{v}_2$ — $\vec{v}_1, \vec{v}_2$ orthonormal, R = radius

β pbal, α assumed pbal

want
(1) $\alpha(0) = \beta(0)$
(2) $\alpha'(0) = \beta'(0) \quad (T_\alpha(0) = T_\beta(0))$
(3) $\alpha''(0) = \beta''(0) \quad (\kappa N_\alpha(0) = \kappa N_\beta(0))$

(1) $\alpha(0) = P + R\vec{v}_1 \qquad \Rightarrow P = \alpha(0) + \frac{1}{\kappa_\alpha(0)}N_\alpha(0)$
(2) $\alpha'(0) = \vec{v}_2 \qquad \Rightarrow v_2 = T_\alpha(0)$
(3) $\underset{\kappa N}{\overset{\shortparallel}{\alpha''(0)}} = -\frac{1}{R}\vec{v}_1 \qquad \Rightarrow \vec{v}_1 = -N_\alpha(0)$, $R = \frac{1}{\kappa_\alpha(0)}$

write κ for $\kappa(0)$, N for $N(0)$, T for $T(0)$, then

$$\Rightarrow \beta(s) = \left[\alpha(0) + \tfrac{1}{\kappa}N(0)\right] + \tfrac{1}{\kappa}\left[\cos\kappa s\,(-N) + \sin\kappa s\, T\right]$$

"osculating circle" in T-N plane thru P.

4.4 $\vec{\alpha}(t)$ rpc. $s(t) = \int_a^t \|\alpha'(r)\|\,dr$ $\qquad s'(t) = \|\alpha'(t)\| > 0$ (ν)

so $s(t)$ has inverse $t(s)$..

$\beta(s) = \alpha(t(s))$

$\beta'(s) = T = \alpha'(t(s))\, t'(s) = \frac{1}{\nu}\alpha'$

$\nu T = \alpha'$

$\frac{d}{ds}$: $\underbrace{\nu_s T}_{0,\ \nu \equiv \text{const}} + \nu \underbrace{T_s}_{= \kappa N} = \alpha''(t(s))\frac{dt}{ds} = \frac{1}{\nu}\alpha''$

$\nu\kappa N = \frac{1}{\nu}\alpha''(t) \Rightarrow \boxed{\alpha''(t) = \nu^2\kappa\vec{N}}$

Newton $\Rightarrow \mu m g \geq m|\alpha''| = m\nu^2\kappa$ (we are looking at force components in the $\vec{N}$ dir)

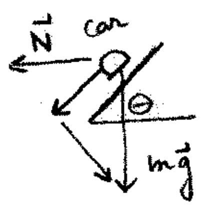
μ ↑ max coeff fric

$\mu g \geq \nu^2\kappa$

$\nu \leq \sqrt{\frac{\mu g}{\kappa}} = \sqrt{\mu g R}$ if $R = \frac{1}{\kappa}$ is radius of curvature.

For a banked road, gravity helps:

cross-section



center of curvature circle

$\swarrow \begin{bmatrix}-\cos\theta\\-\sin\theta\end{bmatrix}$

$\searrow \begin{bmatrix}\sin\theta\\-\cos\theta\end{bmatrix}$

$m\alpha'' = m\nu^2\kappa\vec{N}$

what matters is force // to road surface:

$\cancel{m}\alpha'' \cdot \langle -\cos\theta, -\sin\theta\rangle \leq \cancel{m} g\sin\theta$ ← gravity

$+ \mu(\underbrace{\cancel{m}g\cos\theta}_{\text{grav}} + \underbrace{\cancel{m}\alpha''\cdot\langle -\sin\theta, \cos\theta\rangle}_{\text{centripetal}})$

$\nu^2\kappa\cos\theta \leq g\sin\theta + \mu g\cos\theta + \mu\kappa\nu^2\sin\theta$

$\div\cos\theta$ & collect terms: $\nu^2\kappa(1-\mu\tan\theta) \leq g(\mu + \tan\theta)$

$\nu^2 \leq \frac{g(\mu+\tan\theta)}{1-\mu\tan\theta}$

$V \leq \sqrt{\frac{g(\mu+\tan\theta)}{1-\mu\tan\theta}}$

4.6 For a plane curve,

planar curvature $= \frac{d\theta}{ds}$

One way to derive claimed planar curvature formula:

$T = \beta_s$

$T' = \beta_{ss} = \kappa N$ $\qquad$ if $T = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$

$N = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$ plane curve

$\alpha(t) = \beta(s(t))$ $\qquad \beta$ pbal

$\left.\begin{aligned}\alpha_t &= \beta_s s' \\ \alpha_{tt} &= \beta_{ss}(s')^2 + \beta_s s''\end{aligned}\right\}$ so $\alpha_t \times \alpha_{tt} = (s')^3 \underbrace{\beta_s \times \beta_{ss}}_{\kappa\begin{bmatrix}0\\0\\1\end{bmatrix}}$

so $(\alpha_t \times \alpha_{tt}) \cdot \begin{bmatrix}0\\0\\1\end{bmatrix} = \nu^3\kappa$

i.e. $\begin{vmatrix} 0 & 0 & 1 \\ x' & y' & 0 \\ x'' & y'' & 0 \end{vmatrix} = \kappa v^3$

$$\boxed{\kappa = \frac{x'y''-x''y'}{((x')^2+(y')^2)^{3/2}}}$$

w/o abs vals is planar curvature
w/ them is Frenet curvature

<u>ellipse</u> $\alpha(t) = \begin{bmatrix} a\cos t \\ b\sin t \end{bmatrix}$, $\alpha' = \begin{bmatrix} -a\sin t \\ b\cos t \end{bmatrix}$, $\alpha'' = \begin{bmatrix} -a\cos t \\ -b\sin t \end{bmatrix}$

$$\kappa = \frac{(-a\sin t)(-b\sin t) - (-a\cos t)(b\cos t)}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}} = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}$$

4.7 $\beta(t) = e^t \begin{bmatrix} \cos t \\ \sin t \\ 1 \end{bmatrix}$

using formulas p30-31
& 1/23 notes

$\beta' = e^t \begin{bmatrix} \cos t - \sin t \\ \sin t + \cos t \\ 1 \end{bmatrix}$; $v = |\beta'| = e^t\sqrt{3}$
$s'' = v' = e^t\sqrt{3}$

$$\boxed{T = \frac{\beta'}{v} = \frac{1}{\sqrt{3}} \begin{bmatrix} \cos t - \sin t \\ \sin t + \cos t \\ 1 \end{bmatrix}}$$

$\beta'' = e^t \begin{bmatrix} \cos t - \sin t - \sin t - \cos t \\ \sin t + \cos t + \cos t - \sin t \\ 1 \end{bmatrix}$

$= e^t \begin{bmatrix} -2\sin t \\ 2\cos t \\ 1 \end{bmatrix}$

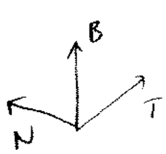
$\beta''' = e^t \begin{bmatrix} -2\sin t - 2\cos t \\ 2\cos t - 2\sin t \\ 1 \end{bmatrix}$

$B = \dfrac{\beta' \times \beta''}{|\beta' \times \beta''|}$

$\beta' \times \beta'' \parallel \begin{vmatrix} i & j & k \\ \cos - \sin & \sin + \cos & 1 \\ -2\sin & 2\cos & 1 \end{vmatrix}$

$= \begin{bmatrix} \sin t - \cos t \\ -\sin t - \cos t \\ 2 \end{bmatrix}$

$$\boxed{B = \frac{1}{\sqrt{6}} \begin{bmatrix} \sin t - \cos t \\ -\sin t - \cos t \\ 2 \end{bmatrix}}$$



$N \parallel B \times T$: $\begin{vmatrix} i & j & k \\ \sin - \cos & -\sin - \cos & 2 \\ \cos - \sin & \sin + \cos & 1 \end{vmatrix} = \begin{bmatrix} -3\sin - 3\cos \\ -3\sin + 3\cos \\ 0 \end{bmatrix}$

$$\boxed{N = \frac{1}{\sqrt{2}} \begin{bmatrix} -\sin t - \cos t \\ -\sin t + \cos t \\ 0 \end{bmatrix}}$$

$$\kappa = \frac{|\beta' \times \beta''|}{|\beta'|^3} = \text{(work on previous page)}$$

$$= \frac{e^{2t}\sqrt{6}}{e^{3t}\, 3\sqrt{3}} = \boxed{\frac{\sqrt{2}}{3e^t}}$$

$$\tau = \frac{(\beta' \times \beta'') \cdot \beta'''}{|\beta' \times \beta''|^2} = \frac{e^{3t}\,(2)}{e^{4t}\, 6} = \boxed{\frac{1}{3e^t}}$$

We checked this with MAPLE the next week!

5.3 Read the discussion on p. 37-38.

The projection of $\beta(s)$ to the initial plane $\perp$ to the "axis" $\vec{u}$ is

$\gamma(s) = \beta(s) - s\cos\theta\, \vec{u}$ where $\vec{u} \cdot \beta' \equiv \cos\theta$, $\vec{u} \cdot \vec{N} \equiv 0$, $\vec{u} \cdot B \equiv \sin\theta$.

If κ, τ are constant,
then the $\exists!$ thm for curves guarantees $\beta(s)$ <u>is</u> a helix going around a real cylinder (w circ cross-sec)

since for $\alpha(t) = \langle a\cos t, a\sin t, bt \rangle$

$\kappa = \frac{a}{a^2+b^2}$, $\tau = \frac{b}{a^2+b^2}$ can attain any real values $\kappa > 0$, $\tau \in \mathbb{R}$ for appropriate a, b.

If $\gamma(s)$ is a circle,
then note that γ <u>is</u> a const speed parameterization,

Since $\gamma' = \beta' - \cos\theta\, \vec{u}$

$\gamma' \cdot \gamma' = 1 - 2\cos^2\theta + \cos^2\theta = \sin^2\theta$

so $|\gamma'| = \sin\theta$.

Thus $\beta(t) = \gamma(t) + [t\cos\theta]\, \vec{u}$ is an a-b helix, with $b = \cos\theta$, a = radius of circle γ.

5.4) $\beta(s) = \left\langle \frac{(1+s)^{3/2}}{3}, \frac{(1-s)^{3/2}}{3}, \frac{s}{\sqrt{2}} \right\rangle$

$\beta' = \left\langle \frac{1}{2}(1+s)^{1/2}, \frac{1}{2}(1-s)^{1/2}, \frac{1}{\sqrt{2}} \right\rangle$

$\beta' \cdot \beta' = \frac{1}{4}(1+s+1-s) + \frac{1}{2} = 1$ so β pbal.

Since $\beta' \cdot \vec{e}_3 \equiv \frac{1}{\sqrt{2}}$, β is a helix.

5.6) I could do this on Maple!

5.7) β pbal.

<u>If</u> $N \cdot \vec{u} \equiv 0$

then $\vec{T} \cdot \vec{u} \equiv$ const (since $(T \cdot u)' = \kappa \vec{N} \cdot \vec{u} \equiv 0$)

So β is a helix

Conversely, if β is a helix

$T \cdot \vec{u} \equiv \cos\theta$ const

$\Rightarrow \kappa \vec{N} \cdot \vec{u} \equiv 0$

$\Rightarrow$ [illegible] $(\kappa \neq 0)$

HW I: Let $\kappa(s), \tau(s)$ fixed on $[0,L]$

Let α solve

$$\begin{cases} \alpha' = T \\ T' = \kappa N \\ N' = -\kappa T + \tau B \\ B' = -\tau N \\ \alpha(0) = \alpha_0 \\ T(0) = T_0 \\ N(0) = N_0 \\ B(0) = B_0 \end{cases}$$

$\{T_0, N_0, B_0\}$ orthonormal positively-oriented frame ($\det = +1$)

Let β solve

$$\begin{cases} \beta' = T \\ T' = \kappa N \\ N' = -\kappa T + \tau B \\ B' = -\tau N \\ \beta(0) = \beta_0 \\ T(0) = T_1 \\ N(0) = N_1 \\ B(0) = B_1 \end{cases}$$

$\{T_1, N_1, B_1\}$ orthonormal positive frame.

Show α and β differ by a rigid motion of $\mathbb{R}^3$,

i.e. $\exists\ \rho(\vec{x}) = \Theta x + \vec{b}$ $\quad \Theta$ a rotation

s.t. $\beta(s) = \rho(\alpha(s))$

pf: $\Theta := \left[T_1 | N_1 | B_1 \right] \left[T_0 | N_0 | B_0 \right]^{-1}$ transforms $\{T_0, N_0, B_0\}$ to $\{T_1, N_1, B_1\}$ and is a rotation (orthog trans with $\det = +1$) because each of our frame matrices has $\det = +1$.

$\rho(\alpha_0) = \beta_0$ iff $\Theta\alpha_0 + b = \beta_0$; $\quad b := \beta_0 - \Theta\alpha_0$

For ρ defined as above define $\gamma(s) = \rho(\alpha(s))$.
We can deduce $\gamma(s) = \beta(s)$ if we show γ solves the IVP for β

Let $\alpha' = T, N, B$ be the frame for α

Define $\tilde{T} := \Theta T$
$\tilde{N} := \Theta N$
$\tilde{B} := \Theta B$

Then

$$\begin{cases} \gamma' = (\Theta\alpha + \vec{b})' = \Theta\alpha' = \tilde{T} \\ \tilde{T}' = (\Theta T)' = \Theta T' = \Theta \kappa N = \kappa \Theta N = \kappa \tilde{N} \\ \tilde{N}' = (\Theta N)' = \Theta N' = \Theta(-\kappa T + \tau B) \\ \qquad = -\kappa \tilde{T} + \tau \tilde{B} \\ \tilde{B}' = (\Theta B)' = \Theta B' = \Theta(-\tau N) = -\tau \tilde{N} \\ \gamma(0) = \beta_0 \text{ by construct} \\ \tilde{T}(0) = \Theta T_0 = T_1 \\ \tilde{N}(0) = \Theta N_0 = N_1 \\ \tilde{B}(0) = \Theta B_0 = B_1 \end{cases}$$

(last three) by construction.

Thus γ solves IVP for β.
Thus $\gamma = \beta$ ∎