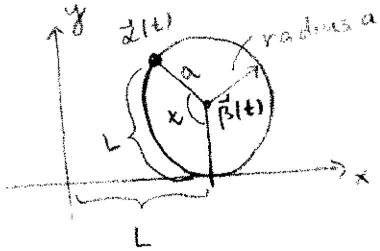
Math 4530
HW set I, Due 1/21

Iabcde : Applications of Pythag thm
Chapter 1:
1.13, 1.14, ~~[illegible]~~, 1.22, 1.25
2.2, 2.7, 2.8, 3.5

1.13



$L = at =$ x-coord of center

let $\vec{\beta}(t) =$ coords of disc center,

so $\vec{\beta}(t) = \begin{bmatrix} at \\ a \end{bmatrix}$

$\vec{\alpha}(t) - \vec{\beta}(t) = \begin{bmatrix} -a\sin t \\ -a\cos t \end{bmatrix}$

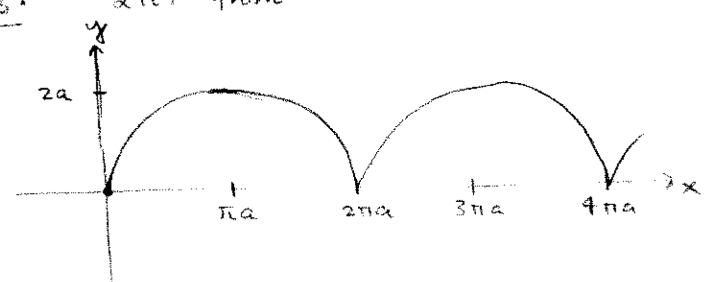
so $\vec{\alpha}(t) = \begin{bmatrix} at \\ a \end{bmatrix} + \begin{bmatrix} -a\sin t \\ -a\cos t \end{bmatrix} = a\begin{bmatrix} t - \sin t \\ 1 - \cos t \end{bmatrix}$ ✓

1.14 $\vec{\gamma}(t) = \underbrace{\begin{bmatrix} A \\ B \end{bmatrix}}_{\text{this translates by } \begin{bmatrix} A \\ B \end{bmatrix}} + a\underbrace{\begin{bmatrix} t - \sin t \\ \cos t - 1 \end{bmatrix}}_{\text{this is reflection of } \vec{\alpha}(t) \text{ across x-axis}}$

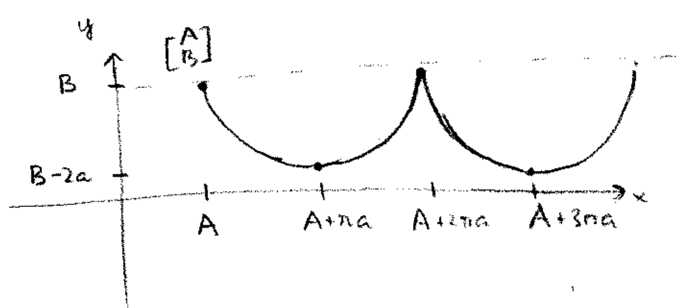
this is reflection of $\vec{\alpha}(t)$ across x-axis [same x-coord, opposite y-coord]

this translates by $\begin{bmatrix} A \\ B \end{bmatrix}$

graphs: $\vec{\alpha}(t)$ from 1.5 :

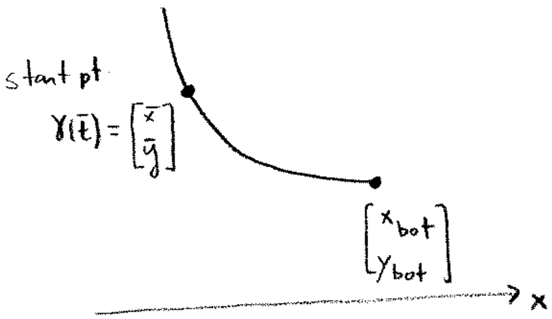


$\vec{\gamma}(t)$ from 1.6:



1.14 cont'd.

this a difficult model.



$$\text{Physics}: \quad \underbrace{\tfrac{1}{2}mv^2}_{KE} + \underbrace{mgy}_{PE} = \text{const} = mg\bar{y}$$

since particle starts at rest.

Thus $\boxed{\tfrac{1}{2}v^2 + g(y-\bar{y}) = 0}$

Since "t" has already been reserved as the parameter in the cycloid, let us temporarily use "s" for time, $0 \le s \le T$

as the particle moves x is an increasing fcn of s,

in fact speed $v = \sqrt{(dx/ds)^2 + (dy/ds)^2}$ $\qquad \frac{dy}{ds} = \frac{dy}{dx}\frac{dx}{ds}$

$$\boxed{\sqrt{2g(\bar{y}-y)}} = v = \sqrt{1+(y')^2}\,\frac{dx}{ds} \qquad y' = \frac{dy}{dx}$$

Physics box

so also s is an increasing fcn of x (1 var inv fcn thm)

$$T = \int_0^T ds = \int_{\bar{x}}^{x_{bot}} \frac{ds}{dx}\,dx = \int_{\bar{x}}^{x_{bot}} \sqrt{\frac{1+(y')^2}{2g(\bar{y}-y)}}\,dx \qquad \left(\frac{ds}{dx} = \frac{1}{(dx/ds)} = \frac{\sqrt{1+(y')^2}}{\sqrt{2g(\bar{y}-y)}}\right)$$

now do another change of variables, from the cycloid parameterization

$$x(t) = A + a(t-\sin t), \qquad y(t) = B + a(\cos t - 1)$$

$$dx = a(1-\cos t)\,dt \qquad dy/dx = \frac{dy/dt}{dx/dt} = \frac{-a\sin t}{a(1-\cos t)} = \frac{\sin t}{1-\cos t}$$

$$\bar{y} - y = B + a(\cos\bar{t} - 1) - [B - a(\cos t - 1)] = a(\cos\bar{t} - \cos t)$$

so

$$\boxed{T = \frac{1}{\sqrt{2g}}\int_{\bar{t}}^{\pi} \sqrt{\frac{1 + \left(\frac{\sin t}{1-\cos t}\right)^2}{a(\cos\bar{t} - \cos t)}}\; a(1-\cos t)\,dt}$$

$$1 - 2\cos t + \cos^2 t + \sin^2 t = 2 - 2\cos t$$

$$T = \sqrt{\frac{a}{2g}} \int_{\bar{t}}^{\pi} \sqrt{\frac{(1-\cos t)^2 + \sin^2 t}{\cos\bar{t} - \cos t}}\,dt$$

$$\boxed{T = \sqrt{\frac{a}{g}} \int_{\bar{t}}^{\pi} \sqrt{\frac{1-\cos t}{\cos\bar{t} - \cos t}}\,dt}$$

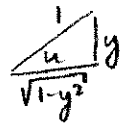
Whew!

$u = \arcsin y$

$\sin u = y$

$(\cos u)\,\frac{du}{dy} = 1$

$\frac{du}{dy} = \frac{1}{\sqrt{1-y^2}}$



$$\frac{d}{dt} \arcsin\left[\frac{\sqrt{2}\cos t/2}{\sqrt{1+\cos\bar{t}}}\right]$$

$$= \frac{1}{\sqrt{1 - \frac{2\cos^2 t/2}{1+\cos\bar{t}}}} \; \frac{\sqrt{2}}{\sqrt{1+\cos\bar{t}}} \left(-\tfrac{1}{2}\sin t/2\right)$$

$$= -\frac{1}{\sqrt{2}} \; \frac{\sin t/2}{\sqrt{1+\cos\bar{t} - 2\cos^2 t/2}}$$

$\cos t = 1 - 2\sin^2 t/2$; $\sin t/2 = \sqrt{\frac{1-\cos t}{2}}$

$= 2\cos^2 t/2 - 1$

$$= -\frac{1}{\sqrt{2}} \; \frac{\sqrt{1-\cos t}\;\frac{1}{\sqrt{2}}}{\sqrt{\cos\bar{t} - \cos t}} = -\frac{1}{2}\sqrt{\frac{1-\cos t}{\cos\bar{t} - \cos t}} \checkmark$$

so, as claimed,

$$T = \sqrt{\frac{a}{g}}\,(-2)\arcsin\left[\frac{\sqrt{2}\cos t/2}{\sqrt{1+\cos\bar{t}}}\right]_{\bar{t}}^{\pi}$$

$$= -2\sqrt{\frac{a}{g}}\left[\arcsin 0 - \underbrace{\arcsin\left[\frac{\sqrt{2}\cos \bar{t}/2}{\sqrt{1+\cos\bar{t}}}\right]}_{=1}\right]$$

$$= 2\sqrt{\frac{a}{g}}\;\pi/4 = \frac{\pi}{2}\sqrt{\frac{a}{g}} \longrightarrow \boxed{T = \frac{\pi}{2}\sqrt{\frac{a}{g}}}$$

1.76 (not assigned)

$\cos 3t = \cos t\,(\cos 2t) - \sin t \sin 2t$

$= \cos t\,(\cos^2 t - \sin^2 t) - 2\sin^2 t \cos t$

$= \cos t\,(\cos^2 t - 3\underbrace{\sin^2 t}_{1-\cos^2 t})$

$= \cos t\,(4\cos^2 t - 3)$

$\cos 3t = -3\cos t + 4\cos^3 t$

so $a\left[\frac{3}{4}\cos t + \frac{1}{4}\cos 3t\right]$

$= a\cos^3 t$ ✓

$\sin 3t = \sin 2t \cos t + \cos 2t \sin t$

$= 2\sin t \cos^2 t + (\cos^2 t - \sin^2 t)\,\sin t$

$= \sin t\,(3\cos^2 t - \sin^2 t)$

$= \sin t\,(3 - 4\sin^2 t)$

so $a\left[\frac{3}{4}\sin t - \frac{1}{4}\sin 3t\right]$

$= a\sin^3 t$ ✓

If $\alpha(t) = \begin{bmatrix} a\cos^3 t \\ a\sin^3 t \end{bmatrix}$ then $x^{2/3} + y^{2/3} = a^{2/3}\left[\cos^2 t + \sin^2 t\right] = a^{2/3}$

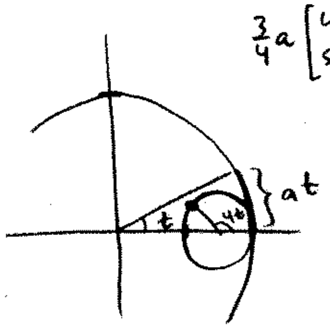
conversely if $x^{2/3} + y^{2/3} = a^{2/3}$

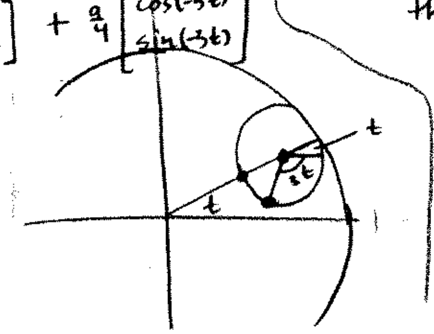
then $\begin{bmatrix} x^{1/3} \\ y^{1/3} \end{bmatrix}$ is on circle of rad $a^{1/3}$

so may write $x^{1/3} = a^{1/3}\cos t$, $y^{1/3} = a^{1/3}\sin t$ ∎

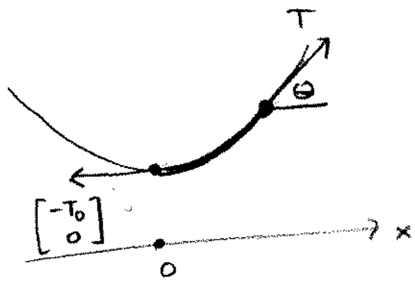
so $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a\cos^3 t \\ a\sin^3 t \end{bmatrix}$

little circle center $\frac{3}{4}a\begin{bmatrix}\cos t \\ \sin t\end{bmatrix} + \frac{a}{4}\begin{bmatrix}\cos(-3t) \\ \sin(-3t)\end{bmatrix}$ displacement from ctr.





1.22



horizontal force balancing: $T \cdot \vec{e}_1 = T_0$

so $\vec{T} = \begin{bmatrix} T_0 \\ T_0 y' \end{bmatrix}$ since $\vec{T} \parallel \begin{bmatrix} 1 \\ y' \end{bmatrix}$

on the other hand, the right end (at "x") is holding up the whole cable piece from the center:

$$\int_0^x W\sqrt{1+(y'(s))^2}\, ds = T_0 y'$$

$$\frac{d}{dx}: \quad W\sqrt{1+y'(x)^2} = T_0 y''$$ ∎

let $z = \frac{dy}{dx}$ then

$$\frac{dz}{dx} = \frac{W}{T_0}\sqrt{1+z^2}$$

$$\frac{dz}{\sqrt{1+z^2}} = \frac{W}{T_0}\, dx$$

$z = \sinh u$

$dz = \cosh u\, du$

$\sqrt{1+z^2} = \cosh u$

$u = \int du = \frac{W}{T_0}x + C$

$z = \sinh u = \sinh\left(\frac{W}{T_0}x + c\right)$

$z(0) = 0$ so

$z = \sinh\left(\frac{W}{T_0}x\right)$

so $y = \int z\, dx = \frac{T_0}{W}\cosh\left(\frac{W}{T_0}x\right) + D$

∎

(1.25) $\vec{\alpha}(t) = \begin{bmatrix} \frac{1}{\sqrt{2}}\cos t \\ \sin t \\ \frac{1}{\sqrt{2}}\cos t \end{bmatrix}$ $\quad 0 \le t \le 2\pi$

$\vec{\alpha}'(t) = \begin{bmatrix} -\frac{1}{\sqrt{2}}\sin t \\ \cos t \\ -\frac{1}{\sqrt{2}}\sin t \end{bmatrix}$ $\quad \|\alpha'(t)\|^2 = 1$

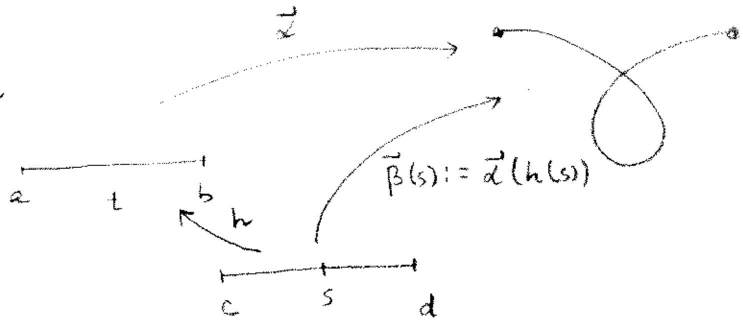
so $L(\alpha) = 2\pi$

$\vec{\alpha}'' = -\vec{\alpha}$

$\vec{\alpha}$ is a circle of radius 1, centered at the origin ($\|\alpha(t)\|^2 = 1$), and lying on the plane $x - z = 0$

(so it is the intersection of this plane thru the origin with the unit sphere.)

2.2



$$L = \int_a^b \|\vec{\alpha}'(t)\|\,dt \qquad \text{if } \begin{array}{l} t = h(s) \\ dt = h'(s)\,ds \end{array}$$

then ✶ $$L = \int_{h^{-1}(a)}^{h^{-1}(b)} \|\vec{\alpha}'(h(s))\|\, h'(s)\,ds$$

Case 1: $h'(s) > 0$, then
$$L = \int_c^d \|\alpha'(h(s))\|\, h'(s)\,ds$$

But $\beta'(s) = \alpha'(h(s))\,h'(s)$
$\|\beta'\| = \|\alpha'\|\,|h'|$

$$= \int_c^d \|\beta'(s)\|\,ds$$

Case 2: $h'(s) < 0$, then
$$L = \int_d^c \|\alpha'(h(s))\|\, h'(s)\,ds = \int_d^c -\|\vec{\beta}'(s)\|\,ds \quad (\text{since } h' < 0)$$
$$= \int_c^d \|\vec{\beta}'(s)\|\,ds \quad (\text{reverse limits})$$

1.2.2 Arclength invariance under reparameterization:

$\alpha : [a,b] \to \mathbb{R}^3$, $\beta(r) : [c,d] \to \mathbb{R}^3$,

$h : [c,d] \to [a,b]$, $h(c) = a$, $h(d) = b$, $h'(r) \geq 0$.

Show $L(\alpha) = L(\beta)$

Pf

$$\int_c^d \|\beta'(r)\|\,dr = \int_c^d \left\| \alpha'(h(r)) \cdot \frac{dh}{ds}(r) \right\| dr$$

this is always non-negative

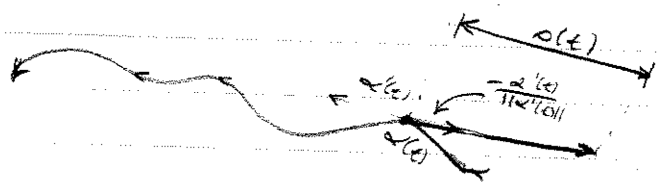
$$= \int_c^d \|\alpha'(h(r))\| \cdot h'(r)\,dr$$

$$= \int_{h(c)}^{h(d)} \|\alpha'(u)\|\,du = \int_a^b \|\alpha'(u)\|\,du$$

1.2.7 Involute curve:

(a) Let $\alpha : [a,b] \to \mathbb{R}^n$. For each $t \in [a,b]$, start at $\alpha(t)$, then move $s(t)$ units in the opposite direction of $\alpha'(t)$;

i.e. $\boxed{I(t) = \alpha(t) - s(t) \dfrac{\alpha'(t)}{\|\alpha'(t)\|}}$

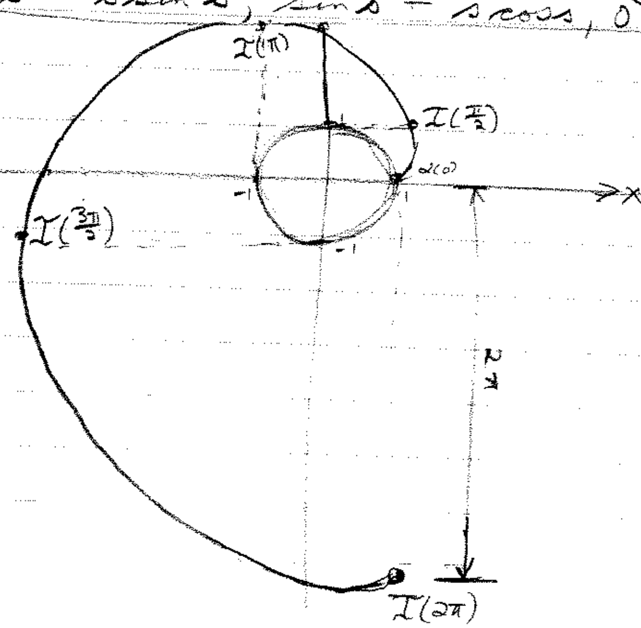


(b) Involute of unit circle $\alpha(t) = \langle \cos t, \sin t, 0 \rangle$.

$\|\alpha'(t)\| = 1 \Rightarrow I(s) = \alpha(s) - s\alpha'(s)$

$\Rightarrow \boxed{I(s) = \langle \cos s + s \sin s, \sin s - s \cos s, 0 \rangle}$

* Involute of the unit circle →



Thanks Ben!

1.2.8 Involute of a helix is a plane curve

Pf

$$I(t) = \alpha(t) - s(t)\frac{\alpha'(t)}{\|\alpha'(t)\|}$$

$$\alpha(t) = \begin{bmatrix} a\cos t \\ a\sin t \\ bt \end{bmatrix} \Rightarrow \alpha'(t) = \begin{bmatrix} -a\sin t \\ a\cos t \\ b \end{bmatrix}$$

$$\Rightarrow \|\alpha'(t)\| = \sqrt{a^2+b^2} = c$$

$$s(t) = \int_0^t c\,dt = ct \Rightarrow I(t) = \alpha(t) - \frac{ct}{c}\alpha'(t)$$

$$\Rightarrow I(t) = \begin{bmatrix} a(\cos t + t\sin t) \\ a(\sin t - t\cos t) \\ 0 \end{bmatrix}$$

$\Rightarrow I(t)$ never leaves the xy-plane.

1.3.5 $\|\vec{v}\times\vec{w}\|^2 = \|\vec{v}\|^2\|\vec{w}\|^2 - (\vec{v}\cdot\vec{w})^2$

$$\text{LHS} = \|\vec{v}\times\vec{w}\|^2 = (\vec{v}\times\vec{w})\cdot(\vec{v}\times\vec{w}) =$$

$$= (v_2w_3 - v_3w_2)^2 + (v_3w_1 - v_1w_3)^2 + (v_1w_2 - v_2w_1)^2 =$$

$$= v_2^2w_3^2 + v_3^2w_2^2 - 2v_2v_3w_2w_3 +$$

$$+ v_1^2w_3^2 + v_3^2w_1^2 - 2v_1v_3w_1w_3 +$$

$$+ v_1^2w_2^2 + v_2^2w_1^2 - 2v_1v_2w_1w_2 =$$

$$= v_1^2(w_2^2+w_3^2) + v_2^2(w_1^2+w_3^2) + v_3^2(w_1^2+w_2^2) - 2v_1v_2w_1w_2 - 2v_1v_3w_1w_3 - 2v_2v_3w_2w_3$$

$$= v_1^2(w_1^2+w_2^2+w_3^2) + v_2^2(w_1^2+w_2^2+w_3^2) + v_3^2(w_1^2+w_3^2)$$

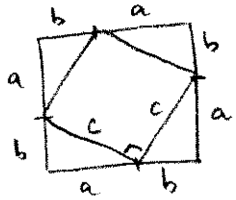
$$- v_1^2w_1^2 - v_2^2w_2^2 - v_3^2w_3^2 - 2v_1v_2w_1w_2 - 2v_1v_3w_1w_3 - 2v_2v_3w_2w_3$$

$$= (v_1^2+v_2^2+v_3^2)(w_1^2+w_2^2+w_3^2) - (v_1w_1 + v_2w_2 + v_3w_3)^2$$

$$= (\vec{v}\cdot\vec{v})(\vec{w}\cdot\vec{w}) - (\vec{v}\cdot\vec{w})^2 = \|\vec{v}\|^2\|\vec{w}\|^2 - (\vec{v}\cdot\vec{w})^2 = \text{RHS} \checkmark$$

Pythag probs.

I a)

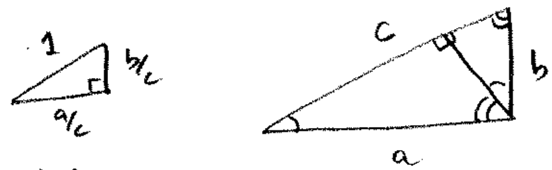


$\text{Area} = (a+b)^2 = a^2 + \cancel{2ab} + b^2$ (square $(a+b)$ by $(a+b)$)

$= c^2 + \cancel{2ab}$ ($c \times c$ square + 4 Δ's with base a, Δ ht b)

thus $a^2+b^2=c^2$!

b)



ref Δ
area A_0

dilate ref Δ by factor of c, to get big Δ

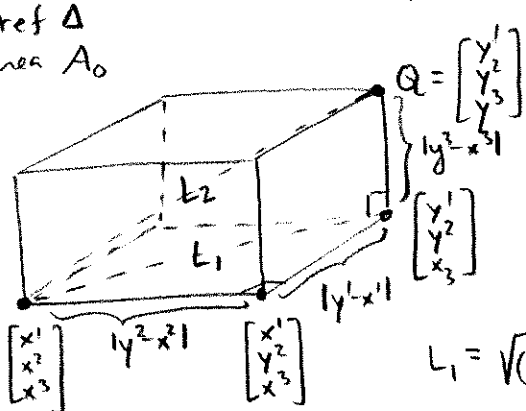
$A = c^2 A_0$

but this Δ is union of two similar Δ's, one scaled by factor of a, other by b, from ref Δ, so also

$A = a^2 A_0 + b^2 A_0$

Hence $a^2+b^2=c^2$!

c)

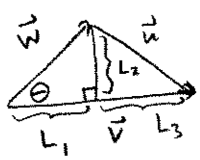


$L_1 = \sqrt{(y^2-x^2)^2 + (y^1-x^1)^2}$

$L_2 = \sqrt{L_1^2 + (y^3-x^3)^2} = \sqrt{(y^1-x^1)^2 + (y^2-x^2)^2 + (y^3-x^3)^2}$

d) Law of cosines

Case I $\theta < \pi/2$. By name choice, assume $\|\vec{w}\| \le \|\vec{v}\|$ so have diagram



$L_1 = \|\vec{w}\|\cos\theta$

$L_3 = \|\vec{v}\| - L_1$

$L_2 = \|\vec{w}\|\sin\theta$

$\|\vec{u}\|^2 = L_2^2 + L_3^2$

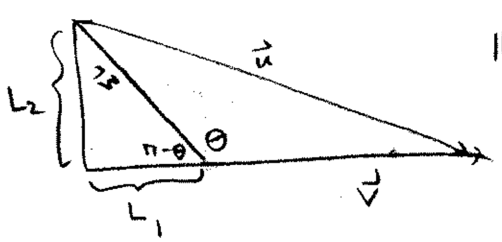
$= \|\vec{w}\|^2 \sin^2\theta + (\|\vec{v}\| - \|\vec{w}\|\cos\theta)^2$

$= \|\vec{w}\|^2(\sin^2\theta + \cos^2\theta) + \|\vec{v}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos\theta$

$= \|\vec{w}\|^2 + \|\vec{v}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos\theta$ ✓

($\theta = \pi/2$ is Pythag thm)

Case II $\theta > \pi/2$,

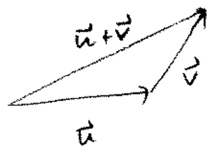


$\|\vec{u}\|^2 = (\|\vec{v}\| + L_1)^2 + L_2^2$

$= (\|\vec{v}\| + \|\vec{w}\|\cos(\pi-\theta))^2 + \|\vec{w}\|\sin(\pi-\theta)^2$

$= \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\|\|\vec{w}\| \underbrace{\cos(\pi-\theta)}_{= -\cos\theta}$ ■

Ie)



$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) \\
&= \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} \\
&= \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 \\
&\le \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2 \quad \text{equality iff } \cos\theta = 1,\ \theta = 0 \\
&= (\|\vec{u}\| + \|\vec{v}\|)^2
\end{aligned}$$

so $\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$

equality iff $\vec{u}, \vec{v}$ are positive scalar multiples.

END!