Math 4530
Wed March 8 §3.4

<u>Gauss' Theorem Egregium</u> : Gauss curvature actually only depends on the first fundamental form, i.e. on knowing the lengths of all tangent vectors! (equivalently, the lengths of all curves)

<u>Discussion</u>:

Let $f: M^2 \to N^2$ be a diffeomorphism (diff'ble with diff'ble inverse).

f is called an <u>isometry</u> iff $f_*: T_pM \to T_{f(p)}N$ is a vector space isometry $\forall p \in M$. This is condition (i) below, equivalent to (ii), (iii), (iv).

(i) i.e. $f_*\vec{v} \cdot f_*\vec{w} = \vec{v}\cdot\vec{w} \quad \forall \vec{v}, \vec{w} \in T_pM, \ \forall p \in M.$

(ii) i.e. $|f_*\vec{v}| = |\vec{v}| \quad \forall \vec{v} \in T_pM \ \forall p \in M$

a linear map which preserves the dot product is equivalent to one which preserves length

(i) $\Rightarrow$ (ii) clear, since
$|f_*v|^2 = f_*v \cdot f_*v$
$|v|^2 = v \cdot v.$

(ii) $\Rightarrow$ (i) since
$$|f_*(v+w)|^2 = (f_*v + f_*w)\cdot(f_*v + f_*w)$$
$$= |f_*v|^2 + |f_*w|^2 + 2 f_*v \cdot f_*w$$

(iii) i.e. $\forall$ curves $\alpha: I \to M$
the length of α = length of $f \circ \alpha$

(ii) $\Leftrightarrow$ (iii)

(ii) $\Rightarrow$ (iii)

$$L(\alpha) = \int_I |\alpha'(t)|\, dt$$

$$L(f\circ\alpha) = \int_I |(f\circ\alpha)'(t)|\, dt$$

but $f_*(\alpha'(t)) := (f\circ\alpha)'(t)$!

and (ii) $\Rightarrow |f_*(v)| = |v|$

so $|(f\circ\alpha)'(t)| = |\alpha'(t)|$ $\blacksquare$

(iii) $\Rightarrow$ (ii)

Let $p \in M$, $v \in T_pM$. Pick $\alpha: I \to M$, $\alpha(0) = p$, $\alpha'(0) = v$.

(iii) $\Rightarrow \displaystyle\int_0^t |\alpha'(r)|\, dr = \int_0^t |(f\circ\alpha)'(r)|\, dr$

FTC ($\frac{d}{dt}$ both sides @ $t=0$) $\Rightarrow |\alpha'(0)| = |(f\circ\alpha)'(0)|$
$|\vec{v}| = |f_*v|$

(iv) Every patch $X: D \to M^2 \subset \mathbb{R}^3$ yields a patch $f\circ X: D \to N^2$ for which the matrices of the first fundamental forms $[I]_X$, $[I]_{f\circ X}$ are <u>identical</u>. $\blacksquare$

(i) $\Rightarrow$ (iv) :

$$\text{(i)} \Rightarrow \begin{bmatrix} X_u\cdot X_u & X_u\cdot X_v \\ X_v\cdot X_u & X_v\cdot X_v \end{bmatrix} = \begin{bmatrix} (f\circ X)_u\cdot(f\circ X)_u & (f\circ X)_u\cdot(f\circ X)_v \\ (f\circ X)_v\cdot(f\circ X)_u & (f\circ X)_v\cdot(f\circ X)_v \end{bmatrix} \Rightarrow [I]_X = [I]_{f\circ X}$$

(iv) ⇒ (i) :

Notice If $\vec{v} = aX_u + bX_v$

$\vec{w} = cX_u + dX_v$

$$\vec{v}\cdot\vec{w} = ac\,(X_u\cdot X_u) + (ad+bc)\,\vec{X}_u\cdot\vec{X}_v + bd\,\vec{X}_v\cdot\vec{X}_v$$

$$= [a, b]\begin{bmatrix} E & F \\ F & G \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix}$$

$f_*\vec{v} = a(f\circ X)_u + b(f\circ X)_v$

$f_*\vec{w} = c(f\circ X)_u + d(f\circ X)_v$

so, assuming (iv)

$$f_*\vec{v}\cdot f_*\vec{w} = [a, b]\begin{bmatrix} E & F \\ F & G \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix}$$

∎

Gauss' Theorem Egregium says isometric surfaces have equal Gauss curvature at corresponding points,

$$K^M(p) = K^N(f(p))$$

even though their shape operators could be quite different!

proof : The proof is to find a formula for K in terms of E, F, G and their derivatives, see (iv) above. The general formula is quite complicated. (see below). We shall derive this formula by equating 3rd order cross partial derivatives for a patch $X(u,v)$! In the special case of an orthogonal patch (i.e. $X_u\cdot X_v = F \equiv 0$), the formula is

$$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right) \qquad I = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}$$

In the even more special case of a conformal or isothermal patch ($F=0$, $E=G$ ⟺ angles in T_pM are same as angles in (u,v) plane)

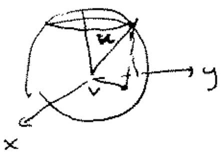
$$K = -\frac{1}{2E}\left(\left(\frac{E_u}{E}\right)_u + \left(\frac{E_v}{E}\right)_v\right) \qquad I = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}$$

(if $E = e^{2\varphi}$ this reduces to

$$K = e^{-2\varphi}\,(\underbrace{\varphi_{uu} + \varphi_{vv}}_{\Delta\varphi})$$
)

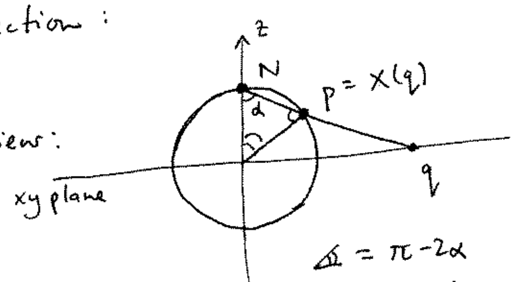
Examples

HW 3.4.6 p137: Use spherical coords $X(u,v) = \langle R\sin u\cos v,\ R\sin u\sin v,\ R\cos v\rangle$ and the page ② formula for K to verify that it yields $K = \frac{1}{R^2}$



Alternately (and we need this later), you can parameterize $S^2 \setminus N$ with inverse stereographic projection:

cross-sectional view:



$\measuredangle = \pi - 2\alpha$

$\cos\alpha = \frac{1}{\sqrt{1+|q|^2}}$

$\sin\alpha = \frac{|q|}{1+|q|^2}$

Exercise Ia: Perhaps using trigonometry and the diagram at right, show inverse stereographic projection is given by

$$\boxed{X(u,v) = \frac{1}{u^2+v^2+1}\langle 2u,\ 2v,\ u^2+v^2-1\rangle}$$

Exercise Ib: By hand or by Maple show that X is a conformal parameterization, i.e.

$$[I]_{\{X_u, X_v\}} = \frac{4}{(1+u^2+v^2)^2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

i.e. $ds = \frac{2}{1+u^2+v^2}\sqrt{du^2+dv^2}$

<u>Remark</u>: One can parameterize (or if you prefer, define) the hyperbolic plane as the unit disc $\{(u,v) \mid u^2+v^2<1\}$, with first fundamental form matrix

$$[I] = \frac{4}{(1-(u^2+v^2))^2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

i.e. $ds = \frac{2}{1-u^2-v^2}\sqrt{du^2+dv^2}$

```
> Kconf:=proc(E1)
  simplify(
  -1/(2*E1(u,v))*(diff(diff(E1(u,v),u)/E1(u,v),u)
  +diff(diff(E1(u,v),v)/E1(u,v),v)),symbolic);
  end:
> Esphere:=(u,v)->4/(1+(u^2+v^2))^2;
  Ehyp:=(u,v)->4/(1-(u^2+v^2))^2;
```

$$Esphere := (u,v) \to \frac{4}{(1+u^2+v^2)^2}$$

$$Ehyp := (u,v) \to \frac{4}{(1-u^2-v^2)^2}$$

```
> Kconf(Esphere);
  Kconf(Ehyp);
```

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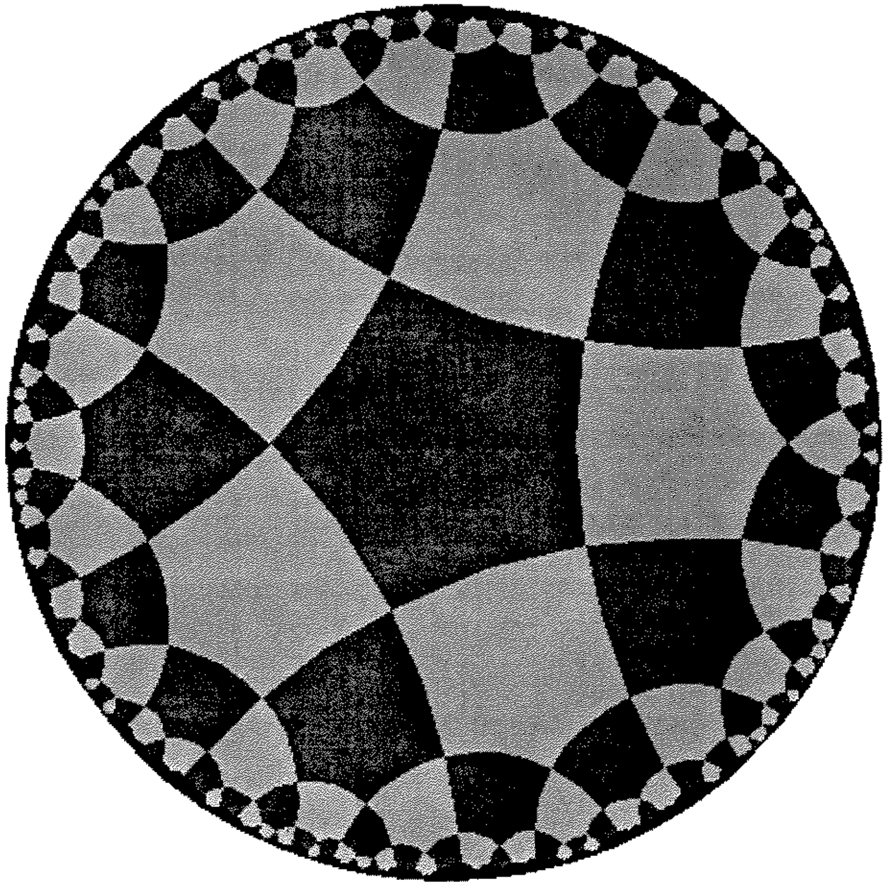
-1

Maple check!

④



An atlas page for the hyperbolic plane is the unit disk, with element of arclength (varying length scale)

$$ds = \frac{2}{1-(u^2+v^2)^2}\sqrt{du^2+dv^2}$$

i.e.

$$[I] = \begin{bmatrix} \frac{4}{(1-(u^2+v^2))^2} & 0 \\ 0 & \frac{4}{(1-(u^2+v^2))^2} \end{bmatrix}$$

Proof of Gauss, in case $[\mathrm{I}] = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}$

follow text. Express 2nd derivs of X & 1st derivs of U in the orthogonal basis $\{X_u, X_v, U\}$

$$X_{uu} = \Gamma^u_{uu} X_u + \Gamma^v_{uu} X_v + \ell U$$
$$X_{uv} = \Gamma^u_{uv} X_u + \Gamma^v_{uv} X_v + m U$$
$$X_{vv} = \Gamma^u_{vv} X_u + \Gamma^v_{vv} X_v + n U$$

this defines the Christoffel symbols. We find them below

know these from II

Also, $U_u = -S(X_u) = -a_{11} X_u - a_{21} X_v$

$U_v \qquad = a_{12} X_u - a_{22} X_v$

Using orthogonality (of $\{X_u, X_v, U\}$)

$$\left.\begin{array}{l} X_{uu} \cdot X_u = \Gamma^u_{uu} E \\ \quad \shortparallel \\ \frac{1}{2}(X_u \cdot X_u)_u = \frac{1}{2} E_u \end{array}\right\} \boxed{\Gamma^u_{uu} = \frac{E_u}{2E}}$$

$$\left.\begin{array}{l} X_{uu} \cdot X_v = \Gamma^v_{uu} G \\ \quad \shortparallel \\ \cancel{(X_u \cdot X_v)_u} - \underbrace{X_u \cdot X_{uv}}_{-\frac{1}{2}E_v} \end{array}\right\} \boxed{\Gamma^v_{uu} = \frac{-E_v}{2G}}$$

$$\left.\begin{array}{l} X_{uv} \cdot X_u = \Gamma^u_{uv} E \\ \quad \shortparallel \\ \frac{1}{2} E_v \end{array}\right\} \Gamma^u_{uv} = \boxed{\frac{E_v}{2E}}$$

everyone else by symmetry!

$$\boxed{\Gamma^v_{vv} = \frac{G_v}{2G}} \quad \boxed{\Gamma^u_{vv} = \frac{-G_u}{2E}} \quad \boxed{\Gamma^v_{uv} = \frac{G_u}{2G}}$$

$$\boxed{\begin{array}{l} X_{uu} = \frac{E_u}{2E} X_u - \frac{E_v}{2G} X_v + \ell U \\ X_{uv} = \frac{E_v}{2E} X_u + \frac{G_u}{2G} X_v + m U \\ X_{vv} = \frac{-G_u}{2E} X_u + \frac{G_v}{2G} X_v + n U \end{array}}$$

$X_{uvu} - X_{uuv} = 0$. In particular the X_v coefficient: (Check! See also text because I skip steps in written work)

$$\frac{E_v}{2E}\left(-\frac{E_v}{2G}\right) + \left(\frac{G_u}{2G}\right)_u + \frac{G_u}{2G}\frac{G_u}{2G} + m(-a_{21})$$
$$-\left[\frac{E_u}{2E}\frac{G_u}{2G} - \left(\frac{E_v}{2G}\right)_v - \frac{E_v}{2G}\frac{G_v}{2G} + \ell(-a_{22})\right] = 0 \,!$$

$\longrightarrow$ But $[S] = [\mathrm{I}]^{-1}[\mathrm{II}]$

so $[\mathrm{I}][S] = [\mathrm{II}]$

$$\begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \ell & m \\ m & n \end{bmatrix}$$

$$\boxed{\begin{array}{l} m = E a_{12} \\ \ell = E a_{11} \end{array}}$$

Thus

$$E\left(a_{11}a_{22}-a_{12}a_{21}\right) = \frac{E_v^2+E_uG_u}{4EG} - \frac{G_u^2+E_vG_v}{4G^2} - \left(\frac{G_u}{2G}\right)_u - \left(\frac{E_v}{2G}\right)_v$$

$$K = \frac{1}{E}\left(\%\right)$$

this is book's formula bottom of page 136

HW 3.4.5 Show K may be rewritten as

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right]$$

(by hand or Maple)