Math 4530

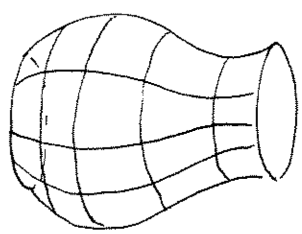
Monday 7 March.

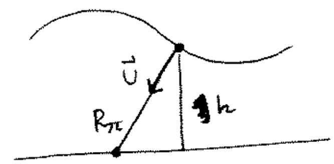
Surfaces of revolution with constant H or K?

Recall, for

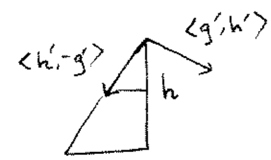
$$X(u,v) = \langle g(u),\, h(u)\cos v,\, h(u)\sin v \rangle$$

$$[S] = \begin{bmatrix} k_\mu & 0 \\ 0 & k_\pi \end{bmatrix} = \begin{bmatrix} \dfrac{h'g'' - g'h''}{((g')^2 + (h')^2)^{3/2}} & 0 \\ 0 & \dfrac{g'}{h\sqrt{g'^2 + h'^2}} \end{bmatrix}$$





$$k_\pi = \frac{1}{R_\pi}.$$



$$\frac{h}{R_\pi} = \frac{g'}{\sqrt{h'^2 + g'^2}}$$

<u>Constant K</u>: Assume $\beta(u) = \langle g(u), h(u) \rangle$ pbal

$$\Rightarrow k_\mu = h'g'' - g'h''$$
$$k_\pi = g'/h$$

$$\Rightarrow K = k_\mu k_\pi = \frac{g'}{h}(h'g'' - g'h'')$$
$$= \frac{g'h'g'' - g'g'h''}{h}$$

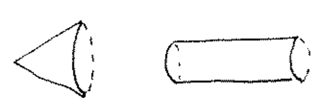
But $g'^2 + h'^2 = 1$
$\Rightarrow \cancel{2}(g'g'' + h'h'') \equiv 0$

$$= \frac{-h'h'h'' - (1 - h'^2)h''}{h}$$

$$\boxed{K = \frac{-h''}{h}}$$ (this was old HW).

$K \equiv 0$ iff $h'' \equiv 0 \Rightarrow h(s) = as + b$
$g(s) = \sqrt{1-a^2}\, s + d$.

<u>cones</u> and <u>cylinders</u>



Scaling in $\mathbb{R}^3$ & effect on K, H:

Given M^2, $R>0$, define $N^2 = RM^2 = \{Rp \text{ s.t. } p \in M^2\}$.
If $X: D \to M^2$ is a patch,
then $Y = RX$ is a patch for N.

$$\begin{aligned} Y_u &= RX_u \\ Y_v &= RX_v \end{aligned} \quad \Rightarrow U^N = U^M.$$

$$\Rightarrow \begin{aligned} U^N_u &= U^M_u \\ U^N_v &= U^M_v \end{aligned}$$

$$\Rightarrow S^N(Y_u) = -U^N_u = -U^M_u = S^M(X_u) = a_{11}X_u + a_{21}X_v \qquad [S^M] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$
$$= \frac{a_{11}}{R} Y_u + \frac{a_{21}}{R} Y_v \, !$$

$$\Rightarrow [S]_{\{Y_u, Y_v\}} = \frac{1}{R}[S]_{\{X_u, X_v\}}.$$

$$\Rightarrow \begin{aligned} K_N &= \frac{1}{R^2} K_M \\ H_N &= \frac{1}{R} K_M \end{aligned}$$

this makes sense if you think about normal curvatures too!

Thus by scaling we may consider the cases

$K = 0, +1, -1$
$H = 0, +1, -1$

and understand the surfaces completely.

$K = +1 \to$ see HW 3.3.13

$K = -1$

$h''(s) - h(s) = 0$

<u>Sol'n</u> $h(s) = A\cosh s + B\sinh s$ solves

$$\begin{cases} h''(s) + h(s) = 0 \\ h(0) = A \\ h'(0) = B. \end{cases}$$

Must have $|h'(0)| < 1$, i.e. $|B| \leq 1$.

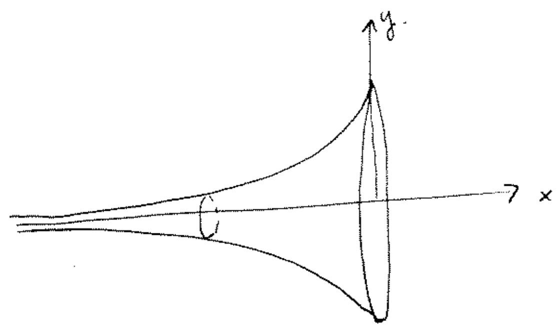
Then $g'(s) = \sqrt{1 - h'(s)^2} = \sqrt{1 - (A\sinh s + B\cosh s)^2}$, until $|A\sinh s + B\cosh s| = 1$.

$$g(s) = \int_0^s \sqrt{1 - (A\sinh t + B\cosh t)}\, dt$$

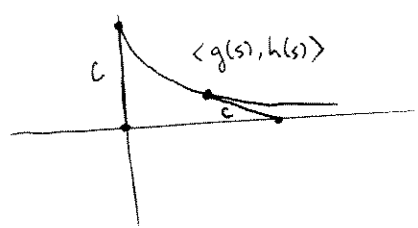
Special cases

pseudosphere: $h(s) = e^s$ $\quad \underline{\underline{s<0}}$ (so $|h'(s)| \le 1$)

$$g(s) = \int_0^s \sqrt{1-e^{2t}}\, dt$$



The profile curve is a tractrix!



Condition for tractrix

$\langle g(s), h(s)\rangle + C\langle g'(s), h'(s)\rangle$ on x-axis

$$h(s) + C h'(s) = 0$$

$$h'(s) = -\frac{1}{C} h(s)$$

$$\Rightarrow h''(s) = -\frac{1}{C} h' = \frac{1}{C^2} h$$

$$\frac{h''(s)}{h(s)} = \frac{1}{C^2} = -K$$

$$K = -\frac{1}{C^2} \;!$$

else, after translation

$$h(s) = A\cosh s$$

$$g(s) = \int_0^s \sqrt{1 - A^2\sinh^2 t}\, dt$$

— See book for picts!

Profile curves for surfaces of constant Gauss curvature
Math 4530
March 7, 2005

It is interesting to find all surfaces of revolution with constant Gauss curvature, and also those with constant mean curvature. The following pictures follow ideas in the text, sections 3.3 and 3.6.

Constant Gauss curvature:
If the profile curve <g(s),h(s)> is pbal (g(s) = component along axis, h(s)>0 is distance to axis), then we derive that

$$K(s) = -h''(s)/h(s).$$

(a) K(s) == 0: We deduce immediately by antidifferentiation and the pbal condition that

$$h(s) = A\,s + B$$

$$g(s) = (1 - A^2)^{\left(\frac{1}{2}\right)}\,s + C$$

In case A = 0, these are cylinders and if A is non-zero these are cones. All cylinders and cones can be created with suitable choice of A and B.

(b) K(s) == -1. In this case the general solution to

$$h''(s) - h(s) = 0$$
$$h(0) = A$$
$$h'(0) = B$$

is

$$h(s) = A\cosh(s) + B\sinh(s).$$

Thus for a pbal curve moving in the positive axis direction,

$$\frac{dg}{ds} = (1 - (A\sinh(s) + B\cosh(s))^2)^{\left(\frac{1}{2}\right)}$$

so, assuming g(0)=0,

$$g(s) = \int_0^s (1 - (A\sinh(t) + B\cosh(t))^2)^{\left(\frac{1}{2}\right)}\,dt$$

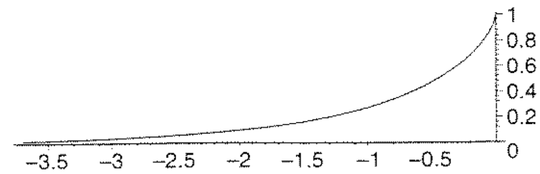
Notice these curves terminate as soon as |dh/ds| becomes larger than 1, since then the pbal condition is lost. Also, if the profile curve as a local minimum, then if we solve the DE starting there the solution is just h(s) = A*cosh(s). If there is never a local minimum then the prototype is $h(s) = \mathbf{e}^s$, say for s<0.

```
> h:=s->exp(s);
  g:=s->int(sqrt(1-exp(2*t)),t=0..s);
```

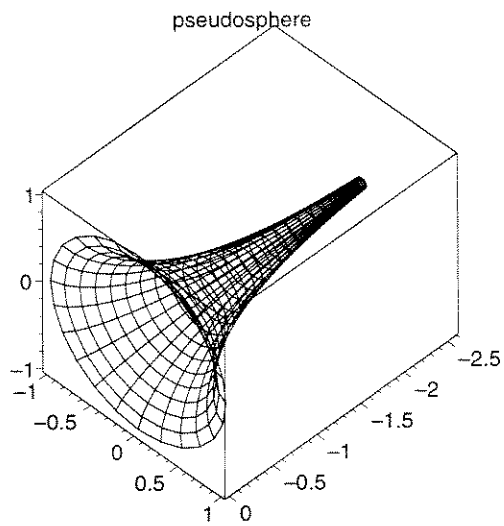
$$h := \exp$$

$$g := s \to \int_0^s \sqrt{1 - \mathbf{e}^{(2\,t)}}\,dt$$

```
> with(plots):
  plot([g(s),h(s),s=-4..0],color=black,scaling=constrained,
  title='pseudosphere profile');
Warning, the name changecoords has been redefined
```




```
> plot3d([g(s),h(s)*cos(v),h(s)*sin(v)],s=-3..0,v=0..2*Pi,
  scaling=constrained, color=black,style=wireframe,
  axes=boxed,title='pseudosphere');
```




```
> h:=s->cosh(s):
  g:=int(sqrt(1-sinh(t)^2),t=0..s):

> plot([g(s),h(s),s=(-.5)..(.5)],color=black,title='another
  profile');
```