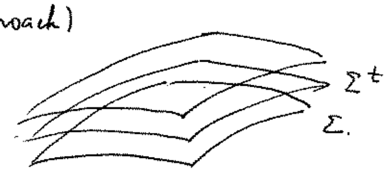
Math 4530
Wed 3/29

We'll make minimal surfaces Friday.
§4.4
Variational approach to mean curvature: (Energy approach)

Set up:



Let Σ be a patch image on M, oriented.

$X : D \to \Sigma$, $\quad \dfrac{X_u \times X_v}{|X_u \times X_v|} = \vec{U}$.

Consider the 1-parameter family of nearby surfaces

$X^t(u,v) = X(u,v) + t\vec{V}(u,v) \qquad \vec{V}$ a vector field on Σ, $\quad |t|$ small

Let $A(t) =$ area of $\Sigma^t = X^t(D)$.

We wish to compute

$$\left.\frac{dA}{dt}\right|_{t=0}$$

$$A(t) = \iint_D |X_u^t \times X_v^t| \, du\, dv$$

$$\begin{aligned} X_u^t \times X_v^t &= (X_u + tV_u) \times (X_v + tV_v) \\ &= X_u \times X_v + t(V_u \times X_v + X_u \times V_v) + O(t^2) \end{aligned}$$

$$\Rightarrow |X_u^t \times X_v^t|^2 = (\ )\cdot(\ ) = |X_u \times X_v|^2 + 2t(X_u \times X_v)\cdot(V_u \times X_v + X_u \times V_v) + O(t^2)$$

$$= |X_u \times X_v|^2 \left(1 + 2t \frac{(X_u \times X_v)}{|X_u \times X_v|^2}\cdot(V_u \times X_v + X_u \times V_v) + O(t^2)\right)$$

$$\Rightarrow |X_u^t \times X_v^t| = |X_u \times X_v|\left(1 + t\frac{\vec{U}}{|X_u \times X_v|}\cdot(V_u \times X_v + X_u \times V_v) + O(t^2)\right)$$

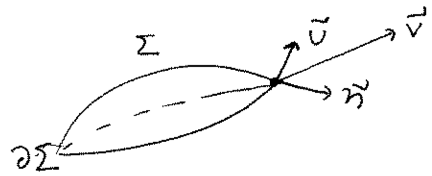
Taylor: $\sqrt{1+z} = 1 + \frac{1}{2}z + O(z^2)$

$$= |X_u \times X_v| + t\,\vec{U}\cdot(V_u \times X_v + X_u \times V_v) + O(t^2)$$

Thus

(1) $$\boxed{\left.\frac{dA}{dt}\right|_{t=0} = \iint_D \vec{U}\cdot(V_u \times X_v + X_u \times V_v)\, du\, dv}$$

This will look better after Green's Thm!



Consider $\displaystyle\int_{\partial\Sigma_1} \vec V\cdot\vec n\, ds$ (because it looks like this measures area growth near $\partial\Sigma$ due to $\vec V$ not being parallel to $\vec U$

Recall $\vec n = \vec T\times\vec U$ along $\partial\Sigma$.

If we parameterize $\partial\Sigma$ as $X(\partial D)$; $\alpha(t) = X(\beta(t))$.

$$\begin{aligned}
&\vec T\times\vec U\, ds \\
&= \frac{\alpha'(t)}{\cancel{|\alpha'(t)|}}\times\vec U\, \cancel{|\alpha'(t)|}\, dt \\
&= (X\circ\beta)'(t)\times\vec U\, dt \\
&= (X_u u' + X_v v')\times\vec U\, dt \\
&= (X_u\, du + X_v\, dv)\times\vec U
\end{aligned}$$

Thus
$$\begin{aligned}
\int_{\partial\Sigma}\vec V\cdot\vec n\, ds &= \int_{\partial D}\vec V\cdot(X_u\times\vec U)\, du + \vec V\cdot(X_v\times\vec U)\, dv \\
&= (\text{Green!}) \iint_D \left(\vec V\cdot(X_v\times\vec U)\right)_u - \left(\vec V\cdot(\vec X_u\times\vec U)\right)_v\, du\,dv \\
&= \left(\begin{array}{l}\text{triple product}\\ \text{rule \& cancellation}\\ \text{of } X_{vu}, X_{uv}\\ \text{terms}\end{array}\right) \iint_D \Big(V_u\cdot(X_v\times\vec U) - \vec V_v\cdot(\vec X_u\times\vec U) \\
&\qquad + \vec V\cdot(\vec X_v\times U_u) - \vec V\cdot(\vec X_u\times\vec U_v)\Big)\, du\,dv \qquad (2)
\end{aligned}$$

these are determinants, and
$$\begin{vmatrix} V_u \\ X_v \\ U\end{vmatrix} = -\begin{vmatrix} U \\ X_v \\ V_u\end{vmatrix} = +\begin{vmatrix} U \\ V_u \\ X_v\end{vmatrix}$$
$$-\begin{vmatrix} V_v \\ X_u \\ U\end{vmatrix} = \begin{vmatrix} U \\ X_u \\ V_v\end{vmatrix}$$

So this term exactly matches (1) on page 1

$$\begin{aligned}
&\vec X_v\times\vec U_u \\
&= \vec X_v\times(-a_1^1 X_u - a_1^2 X_v) \\
&= +a_1^1\, X_u\times X_v \\
&- X_u\times U_v \\
&= +a_2^2\, X_u\times X_v
\end{aligned}$$

So this term
$$\begin{aligned}
&= 2H\ \vec V\cdot(\vec X_u\times\vec X_v)\, du\,dv \\
&= 2H\,\vec V\cdot\vec U\, dA
\end{aligned}$$

Combining (1), (2):

$$(3)\quad \boxed{\left.\frac{dA}{dt}\right|_{t=0} = \int_{\partial\Sigma}\vec V\cdot\vec n\, ds - \iint_\Sigma 2H\,\vec V\cdot\vec U\, dA}$$

If $\vec V$ is defined on all of M, then decompose M into patches and add the results. Interior $\partial\Sigma$ contributions cancel, get

$$(4)\quad \boxed{\left.\frac{dA}{dt}\right|_{t=0} = -\iint_M 2H\,\vec V\cdot\vec U\, dA + \int_{\partial M}\vec V\cdot\vec n\, ds}$$

(if M has boundary)

Corollary

$$\left.\frac{dA}{dt}\right|_{t=0} = 0 \text{ for all } \vec{V} \quad (\text{s.t. } \vec{V} = \vec{0} \text{ on } \partial M)$$

if and only if $H \equiv 0$

$\Leftarrow$ clear

$\Rightarrow$ If $H \neq 0$ near P make a vector field $\vec{V} = \varphi \vec{U}$ where $\varphi > 0$ near p, $\equiv 0$ farther away

$\Rightarrow \frac{dA}{dt} \neq 0$

This is the reason minimal surfaces have their name

<u>But</u> notice, this is only equivalent to M being a <u>critical point</u> for area, not a minimum.

<u>Examples</u> : (Exercise 4.3.5 for next week)

if the graph $z = f(x,y)$ is a minimal surface, then this surface <u>is</u> minimizing (least area) among all comparison graphs with the same boundary values.

- the proof is a neat application of the divergence theorem, see 4.3.5 for hints.

<u>Def</u> $(M, \partial M)$ is <u>stable</u> if $\left.\frac{d^2A}{dt^2}\right|_{t=0} > 0$ for all $\vec{V}$ as above, $\vec{V} = \vec{0}$ on ∂M.

(analogous to a local min in calculus)

<u>Def</u> $(M, \partial M)$ is minimizing if M has the least area among all comparison surfaces with the same area

<u>Example</u> (See Exercise 4.3.6 for next week)

The catenoid, as a surface spanning two congruent circles

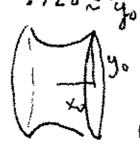
circles close

catenoid

$\frac{x_0}{y_0} \lesssim .528$

Catenoid minimizing

$.528 \lesssim \frac{x_0}{y_0} \lesssim .663$



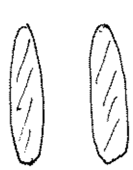
Catenoid local min (stable)

$\frac{x_0}{y_0} > .663$

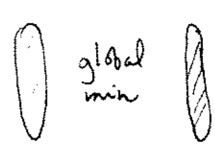
$\nexists$ Catenoid

2 disks

don't minimize, but local min.



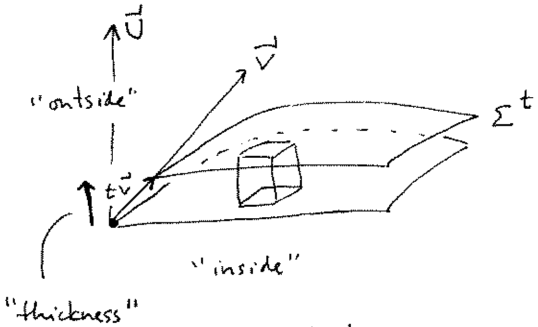
global min.



global min

How does "enclosed" volume change under the variations $X^t(u,v) = X(u,v) + tV$?
(consider $\vec{U}$ to be "exterior" normal).

This is a much easier calculation, and can be done with differentials



$= \text{comp}_{\vec{U}}\, t\vec{V} = t\,\vec{V}\cdot\vec{U}$

So (thinking in the $\vec{U}$ direction), at time t,

$$dVol \approx \underset{\text{area}}{(dA)}\,\underset{\text{height}}{(t\,\vec{V}\cdot\vec{U})}$$

So
$$Vol(t) = Vol(0) + \iint_{\Sigma} t\,\vec{V}\cdot\vec{U}\,dA + O(t^2).$$

$$\boxed{\left.\frac{dVol}{dt}\right|_{t=0} = \iint_M \vec{V}\cdot\vec{U}\,dA}$$

(In fact, thats why integrals of this form are called flux integrals!)

Theorem: M as constant mean curvature $H = -\frac{p}{2\sigma}$ if and only if

$$\left.\frac{d}{dt}\Big[\sigma A(t) - p\,Vol(t)\Big]\right|_{t=0} = 0 \quad \forall \text{ vector fields } \vec{V} \text{ on } M$$

(This is the potential energy formulation of our Newton's law considerations, in the case of constant σ and p)

proof: the expression above

$$= \iint_M \sigma(-2H\,\vec{V}\cdot\vec{U}) - p\,\vec{V}\cdot\vec{U}\,dA$$

(If M has bdry we assume $\vec{V}\equiv 0$ there)

$$= \iint_M (2H\sigma + p)(-\vec{V}\cdot\vec{U})\,dA.$$

∎ since $\vec{V}$ is arbitrary.