Math 4530
Monday 28 March

Because of my diversion into balloons, the only HW due this Friday is Friday's class exercise, 4.2.3, 4.2.8

Balloon science cont'd.

I missed an internal force in my Balloon science derivation, last Friday!
(This is only relevent when σ (surface tension) is not constant, i.e. balloons vs. bubbles).

σ = force/length = energy/area

The correct theorem is

pressure pushing σ

an internal tangential force, trying to push the balloon to region of smaller σ.

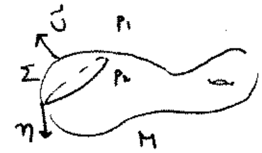
Macroscopic (Newton) I

$$\boxed{\iint_{\Sigma_1} p\vec{U}\, dA - \iint_{\Sigma_1} \nabla_\Sigma \sigma\, dA + \int_{\partial\Sigma_1} \sigma\vec{\eta}\, ds = 0}$$

$\forall$ test subsurfaces Σ_1

surface tension along $\partial\Sigma$

$p = p_2 - p_1$



iff

Microscopic II $\boxed{H = -\frac{p}{2\sigma} \text{ pointwise on } M}$

What is the surface gradient, $\nabla_\Sigma \sigma$?

If $\varphi: M^2 \to \mathbb{R}$ $(\nabla_M \varphi)(p) \in T_pM$ is the unique tangent vector so that

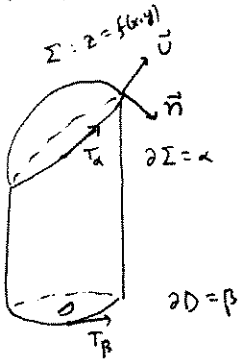
$$V(\varphi) = \nabla_M \varphi(p) \cdot \vec{V} \quad \forall\ \vec{V} \in T_pM$$

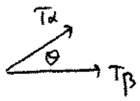
the dir. deriv of φ in dir. of $\vec{V}$

If $\{\vec{e}_1, \vec{e}_2\}$ is an orthonormal basis of T_pM,
then $\nabla_M \varphi = \vec{e}_1(\varphi)\vec{e}_1 + \vec{e}_2(\varphi)\vec{e}_2$ (just like in Euclidean space).

(See page 2 Friday).

We considered a graphical patch.





$T_\alpha = \cos\theta \, T_\beta + \sin\theta \, \vec{e}_3$

$$\iint_\Sigma p \vec{U}\cdot\vec{e}_3 \, dA \; - \; \iint_\Sigma \nabla_\Sigma \sigma \cdot \vec{e}_3 \, dA \; + \int_{\partial\Sigma} \sigma \vec{n}\cdot\vec{e}_3 \, ds \; = \vec{0}\cdot\vec{e}_3 = 0$$

$$\iint_\Sigma p \vec{U}\cdot\vec{e}_3 \, dA = \iint_D p \, dxdy$$

$$\iint_\Sigma \nabla_\Sigma \sigma \cdot \vec{e}_3 \, dA \;\; \text{can show! } * \;\; -\iint_D \nabla\sigma \cdot \frac{\nabla f}{\sqrt{1+|\nabla f|^2}} \, dxdy$$

$\int_{\partial\Sigma} \sigma \vec{n}\cdot\vec{e}_3 \, ds$ $\parallel$ ← mostly done Friday

$$\iint_D \operatorname{div}\left(\sigma \frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) dxdy = 0$$

$$= \nabla\sigma \cdot \frac{\nabla f}{\sqrt{1+|\nabla f|^2}} + \sigma \underbrace{\operatorname{div}\left(\frac{\nabla f}{\sqrt{\;}}\right)}_{2H}$$

exactly cancels

this term caused an inconsistency once we deduced $p + 2\sigma H = 0$.

In the correct proof of the correct theorem, after you cancel the $\nabla\sigma \cdot \frac{\nabla f}{\sqrt{\;}}$ terms

you deduce $\iint_D p + 2H\sigma \, dxdy = 0 \quad \forall D$

$\Rightarrow p + 2H\sigma \equiv 0.$

So that I $\Rightarrow$ II.

Showing II $\Rightarrow$ I proceeds as indicated on Friday: The identity holds on super-graphical Σ, then decompose a large Σ into supergraphical pieces.

$*$ At a point in D

choose $\vec{e}_1 = \frac{\nabla f}{|\nabla f|}$, to compare

$$\nabla_\Sigma \sigma \cdot \vec{e}_3 \, dA \overset{?}{=} \nabla_{\mathbb{R}^2}\sigma \cdot \frac{\nabla f}{\sqrt{1+|\nabla f|^2}} \, dxdy :$$

$$\underbrace{\nabla_\Sigma \sigma \cdot \vec{e}_3}_{} \; dA$$

$$\vec{V}(\sigma) \underbrace{\vec{V}\cdot\vec{e}_3}_{} \underbrace{dA}_{}$$

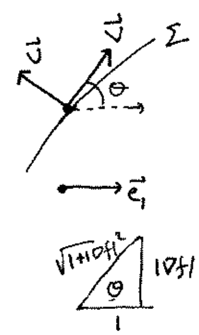
$$\parallel \qquad \parallel \qquad \parallel$$

$$(\cancel{\cos\theta}\, \sigma_x) \; \underbrace{\sin\theta}_{} \; \frac{1}{\cancel{\cos\theta}} dxdy$$

$$\sigma_x \frac{|\nabla f|}{\sqrt{1+|\nabla f|^2}} \, dxdy$$

$$\parallel$$

$$\nabla_{\mathbb{R}^2}\sigma \cdot \frac{\nabla f}{\sqrt{1+|\nabla f|^2}} \, dxdy.$$

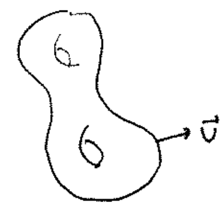


$\vec{V} = \cos\theta \, \vec{e}_1 + \sin\theta \, \vec{e}_3$

The macroscopic force balancing equation leads to interesting interpretations, but first we need a lemma

<u>Theorem</u>: Let M^2 be a compact piecewise regular surface, bounding a region $\mathcal{R}$ in $\mathbb{R}^3$. Let $\vec{\upsilon}$ be the unit exterior normal.

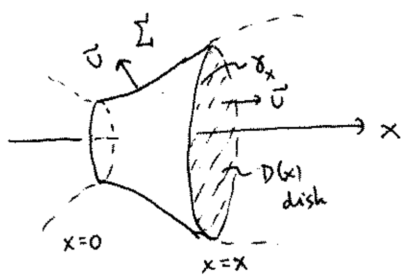
Then $\displaystyle\iint_M \vec{\upsilon}\, dA = \vec{0}$



<u>proof</u>. Let $\vec{W}$ be any constant vector.

$$\left(\iint_M \vec{\upsilon}\, dA\right)\cdot \vec{W} = \iint_M (\vec{\upsilon}\cdot\vec{W})\, dA$$
$$= \iiint_{\mathcal{R}} \underbrace{\operatorname{div}\vec{W}}_{=0}\cdot dV = 0. \quad \blacksquare$$

<u>Example</u> Delaunay surfaces and catenoid revisited. $\left(\begin{matrix} p = \text{const} \\ \sigma = \text{const}, \end{matrix} \ \begin{matrix} \sigma = 0 \\ H = 0\end{matrix}\right)$

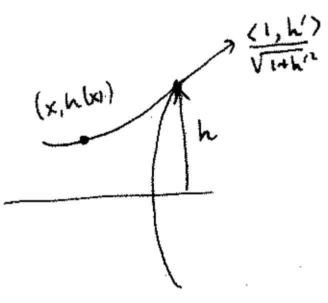


$$p\iint_\Sigma \vec{\upsilon}\, dA + \sigma\int_{\partial\Sigma} \vec{n}\, ds = \vec{0}$$

$$-p\iint_{D(x)} \vec{\upsilon}\, dA + \sigma\int_{\gamma_x} \vec{n}\, ds = p\iint_{D(0)} \vec{\upsilon}\, dA - \sigma\int_{\gamma_0} \vec{n}\, ds$$

profile curve $\langle x, h(x)\rangle$

$= \text{const}$ (no x dependence)
$=$ applied force at a virtual cut at x.



net force vector points in x-dir.
It's component is

$$\boxed{\sigma\left(\frac{2\pi h}{\sqrt{1+h'^2}}\right) - p\pi h^2 = C}$$

this is a "first integral" of the 2^{nd} order ODE for constant mean curvature

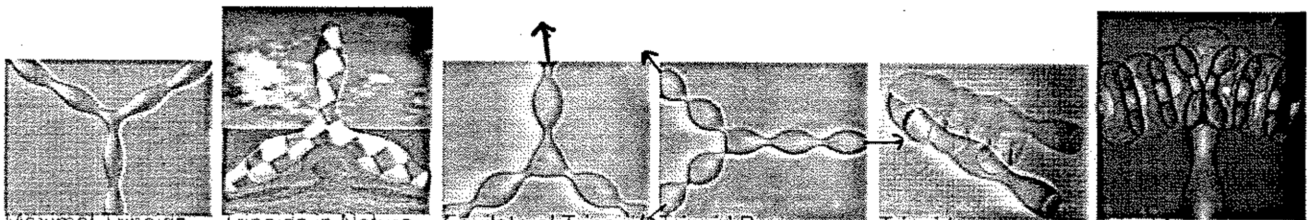
<u>Case 1</u> $p = 0$ $\quad \dfrac{h}{\sqrt{1+h'^2}} = C \quad \Rightarrow h = D\cosh\left(\dfrac{x-x_0}{D}\right)$ Catenoid!

<u>Case 2</u> $p > 0$ get regular Delaunay : $|h'|$ depends on h leads to periodic nature

$p < 0$ get immersed Delaunay.

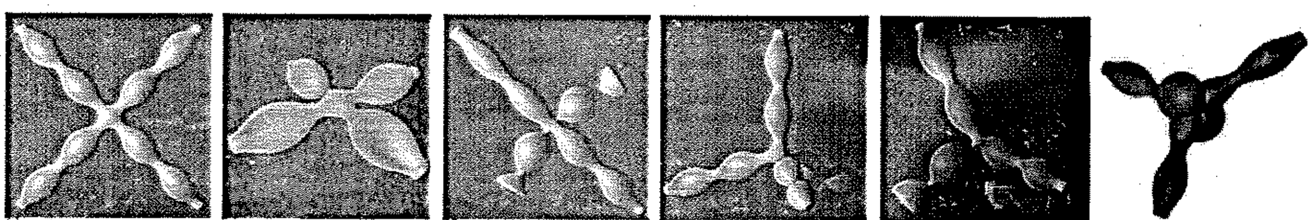
Force balancing in modern day examples: (constant mean curvature)

Trinoids



Maximal Trinoids | Trinoids in Nature | Equilateral Trinoids | Trinoid Pump | Trinoids ++- | Trinoids +--

K-noids



Coplanar | Coplanar | Coplanar | Noncoplanar | Noncoplanar | Noid + bubble

Great project idea: Model balloon shapes, for the rotationally symmetric case

→ figure out how you think σ might depend on radius. (I have ideas).