Math 4530
Fri 25 March

HW: Class exercise today
also, 4.2.3, 4.2.8, 4.3.5, 4.3.6
4.4.2, 4.4.3, 4.4.8

due next Fri. 4/1

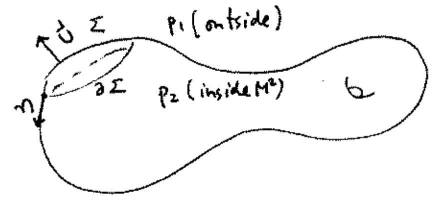
Mean curvature, balloons & bubbles (We're starting chapter 4)

Consider a surface which is held in equilibrium (not moving) by a combination of <u>pressure</u> (pushing the surface) and <u>surface tension</u> (trying to shrink the surface)

Specifically, if Σ is a piece of the surface M^2, the net forces on Σ must be $\vec{0}$:

$$\mathrm{I} \qquad \boxed{\iint_{\Sigma} p\vec{U}\,dA + \int_{\partial\Sigma} \sigma\vec{\eta}\,ds = \vec{0}}$$

for each test piece Σ



p = pressure ($= p_2 - p_1$ see below)
= force/unit area of surface, in normal direction

σ = surface tension = force/unit length along $\partial\Sigma$, in conormal $\vec{\eta}$ direction

e.g. for an inflated balloon p is constant but σ varies, depending on how stretched the surface is.
for a soap bubble p and σ are constant
for a soap film (with no "inside") $p = 0$, σ is constant

<u>Theorem</u> The macroscopic force balancing condition above is <u>equivalent</u> to the differential geometry (microscopic) condition on M^2:

$$\mathrm{II} \qquad \boxed{H = -\frac{p}{2\sigma}}$$

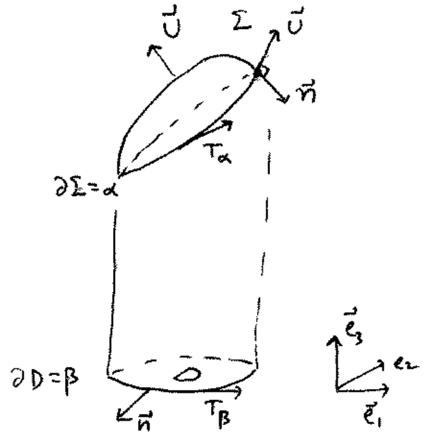
H = mean curvature wrt $\vec{U}$ as chosen above
(so $H = \frac{p}{2\sigma}$ with respect to $-\vec{U}$ = "inner" normal).

So if $p = 0$ surface is minimal ($H \equiv 0$)
if p, σ are non-zero constants, surface has constant mean curvature.

(this is an example of how a physics law leads to a P.D.E. & vice versa)

[Actually, I will show $\mathrm{I} \Rightarrow \mathrm{II}$.
Assuming p, σ are constant I will show $\mathrm{II} \Rightarrow \mathrm{I}$]

<u>proof</u>: begin with a small, graphical piece Σ_1 to show force balancing $\Rightarrow H = \frac{-p}{2\sigma}$



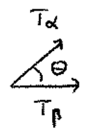
Σ_1 = graph $z = f(x,y)$ above D.

$\vec{U} = \langle -f_x, -f_y, 1\rangle \frac{1}{\sqrt{1+|\nabla f|^2}}$

$$\left(\iint_\Sigma p\vec{U}\right)\cdot \vec{e}_3 \overset{dA_\Sigma}{=} \iint_\Sigma p\,\vec{U}\cdot\vec{e}_3 \; dA_\Sigma$$

$$= \iint_D p \frac{1}{\cancel{\sqrt{1+|\nabla f|^2}}} \cancel{\sqrt{1+|\nabla f|^2}}\; dxdy$$

$$= \iint_D p \; dxdy \quad (*)$$



$T_\alpha = \cos\theta\, T_\beta + \sin\theta\, e_3$

$$\int_{\partial\Sigma_1 = \alpha} \sigma\,\vec{\eta}\cdot\vec{e}_3 \, ds$$

$\|$ ① (see below)

$$\int_{\partial D = \beta} -\sigma\,\vec{U}\cdot\vec{n}\; ds_\beta = \int_\beta +\sigma\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\cdot\vec{n}\, ds$$

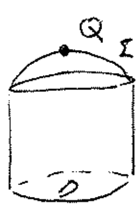
$$(**) \quad = \iint_D \operatorname{div}\left(+\sigma\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) dxdy$$ divergence thm!

$$(*),(**) \Rightarrow \iint_D p + \underbrace{\operatorname{div}\left(\frac{\sigma\nabla f}{\sqrt{1+|\nabla f|^2}}\right)} \; dxdy = 0$$

$$+\left[\nabla\sigma\cdot\frac{\nabla f}{\sqrt{\ }} + \sigma\,\underbrace{\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)}_{\| ②\; 2H}\right)$$

this is HW: show $\left(\frac{f_x}{\sqrt{1+|\nabla f|^2}}\right)_x + \left(\frac{f_y}{\sqrt{1+|\nabla f|^2}}\right)_y$

$= \text{trace}\,(g^{ij}h_{ij})$ for a graph.



If we focus at $Q \in M$ and graph above tangent plane, restrict to D = disk of radius r, and let $r \to 0$ in

$$\frac{1}{\pi r^2}\iint_{D_r} p - \frac{\nabla\sigma\cdot\nabla f}{\sqrt{\ }} + 2\sigma H \; dxdy = 0$$

Deduce $\boxed{p + 2\sigma(Q)H(Q) = 0}$, which finishes $\Rightarrow$. ∎

①: Fun with cross products: $\eta = T_\alpha \times U$ so $\vec{e}_3\cdot\vec{\eta} = \begin{vmatrix} \vec{e}_3 \\ T_\alpha \\ U \end{vmatrix} = \begin{vmatrix} \vec{e}_3 \\ \cos\theta\, T_\beta + \sin\theta\,\vec{e}_3 \\ U \end{vmatrix} = \cos\theta \begin{vmatrix} \vec{e}_3 \\ T_\beta \\ U \end{vmatrix} = -\cos\theta \begin{vmatrix} U \\ T_\beta \\ e_3 \end{vmatrix}$

whereas $\vec{n} = T_\beta \times \vec{e}_3$ so $\vec{U}\cdot\vec{n} = \begin{vmatrix} U \\ T_\beta \\ e_3 \end{vmatrix}$ ← also $ds_\beta = \cos\theta\, ds_\alpha$.

$\Longrightarrow \vec{e}_3\cdot\vec{\eta}\, ds_\alpha = -\cos\theta\,\vec{U}\cdot\vec{n}\, ds_\alpha = -\vec{U}\cdot\vec{n}\, ds_\beta$ ∎

$\Longleftarrow$: If p, σ both constant, and if $H = \frac{-p}{2\sigma}$ $(p + 2\sigma H = 0)$, then for any set-up as on page 2, begin with

$$\iint_D p + 2\sigma H \, dx\,dy = 0$$

work backwards:

$$\iint_D 2\sigma H \, dx\,dy = \sigma \iint_D \operatorname{div}\left(\frac{Df}{\sqrt{1+|Df|^2}}\right) dx\,dy$$

$$= \sigma \int_{\partial D} -\vec{U}\cdot\vec{n} \, ds_\beta$$

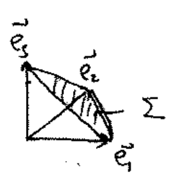
$$= \sigma \int_{\partial \Sigma} \vec{\eta}\cdot\vec{e}_3 \, ds_\alpha$$

$$\iint_D p \, dx\,dy = \iint_\Sigma p \, \vec{U}\cdot\vec{e}_3 \, dA$$

Thus, for a piece of surface which is graphical with respect to a vector $\vec{v}$ (= "$\vec{e}_3$"),

$$\iint_\Sigma p\vec{U}\cdot\vec{v} \, dA + \int_{\partial\Sigma} \sigma\vec{\eta}\cdot\vec{v} \, ds = 0.$$

For a (small enough) piece which is graphical wrt an orthonormal basis $\vec{e}_1, \vec{e}_2, \vec{e}_3$
(i.e. wrt each of these!)

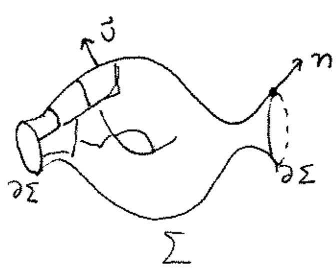


we deduce the vector equality

$$\iint_\Sigma p\vec{U} \, dA + \int_{\partial\Sigma} \sigma\vec{\eta} \, ds = 0$$

For a big piece of surface, decompose it into a partition of special graphical pieces as above, and note that the "interior" co-normal integrals cancel out, to again deduce the macro-scopic version!

$$\iint_\Sigma p\vec{U} \, dA + \int_{\partial\Sigma} \sigma\vec{\eta} \, ds = 0$$



Balloon science: 

1. Explain why our clown balloons begin inflating where they do, and why they grow the way they do as inflated?

2. Why can't you bend a clownballoon too far into a torus shape without crinkling it?

3. aneurysms can be bad news!

4. almost Delaunay clown balloons?