Math 4530
Monday 21 March §3.4–3.5

Recall, we've been discussing Gauss' Theorem Egregium
and its relation to equality of mixed partials for a patch X. (See page 2)

$$X_{,ij\ell} = X_{,i\ell j}$$

$X_{,k}$ components match:

$$\Rightarrow \boxed{-a^s_j a^k_\ell + a^s_\ell a^k_j = g^{si}\left[\Gamma^k_{i\ell,j} - \Gamma^k_{ij,\ell} + \Gamma^r_{i\ell}\Gamma^k_{rj} - \Gamma^r_{ij}\Gamma^k_{r\ell}\right]}$$

$\begin{matrix} j=k=1 \\ s=\ell=2 \end{matrix}$ or $\begin{matrix} j=k=2 \\ s=\ell=1 \end{matrix}$ yields $K = ?$ Gauss Equation

(if 3 or 4 indices agree LHS $=0$ $=$ RHS ?)

U components match:

$$\boxed{\Gamma^k_{ij} h_{k\ell} + h_{ij,\ell} = \Gamma^k_{i\ell} h_{kj} + h_{i\ell,j}}$$

$$\begin{matrix} i=j=1 \\ \ell=2 \end{matrix} \Rightarrow \quad \Gamma^k_{11} h_{k2} + h_{11,2} = \Gamma^k_{12} h_{k1} + h_{12,1}$$

$$\begin{matrix} i=j=2 \\ \ell=1 \end{matrix} \Rightarrow \quad \Gamma^k_{22} h_{k1} + h_{22,1} = \Gamma^k_{21} h_{k2} + h_{21,2}$$

Codazzi-Mainardi.

(all others redundant or vacuous.)

Bonnet's Theorem (analog of $\exists!$ for Frenet System).

Let $[g_{ij}]$, $[h_{ij}]$ be given matrix functions on a simply connected domain D ($\subset \mathbb{R}^2$ or $\mathbb{R}^n$).
assume each is C^∞, symmetric, and that $[g_{ij}] > 0$
($v^i v^j g_{ij} > 0 \ \forall \vec{v} \neq \vec{0}$).

Define $[a^i_j] = [g^{i\ell} h_{\ell j}]$.

Define $\Gamma^k_{ij} = \frac{1}{2} g^{k\ell}(g_{\ell i,j} + g_{\ell j,i} - g_{ij,\ell})$ [This is the general formula for Christoffel symbols, see page 3].

Then if the Gauss & C-M eqtns hold,

$\exists$ $X : D \to \mathbb{R}^3$ ($\mathbb{R}^{n+1}$)
s.t. $[g_{ij}], [h_{ij}], [a^i_j]$ are the matrices of 1^{st}, 2^{nd} ~~fdl~~ forms, & ~~the~~ shape operator.
X is unique up to rigid motion.

idea: One considers the 1^{st} order system of PDE's

$$\begin{cases} X_{,i} = Y_i & i=1,2 \\ Y_{i,j} = \Gamma^k_{ij} Y_k + h_{ij} U & i=1,2,\ j=1,2 \quad \left(\vec{U} := \dfrac{Y_1 \times Y_2}{|Y_1 \times Y_2|}\right) \\ X(q) = p & \\ Y_i(q) = Y^0_i & \text{where } Y^0_i \cdot Y^0_j = g_{ij}(q) \end{cases}$$

Then Gauss & C-M are the formal cross partial conditions which allows one to solve this first order system (research Frobenius Thm).

Re-derivation of Gauss' Eqtn:

This is page 2 from Friday, except with three fixed mathos – see *'s.

$$X_{,ij} = \Gamma^k_{ij} X_{,k} + h_{ij} U$$

$$X_{,ij\ell} - X_{,i\ell j} = 0 \,!$$

$$X_{,ij\ell} = \Gamma^k_{ij,\ell} X_{,k} + \Gamma^k_{ij} X_{,k\ell} + h_{ij,\ell} U + h_{ij} U_{,\ell}$$

$$\boxed{= \Gamma^k_{ij,\ell} X_{,k} + \Gamma^k_{ij}\left\{(\Gamma^r_{k\ell} X_{,r} + \overset{*}{h_{k\ell}} U)\right\} + h_{ij,\ell} U - h_{ij} a^k_\ell X_{,k}}$$

rewrite $\Gamma^r_{ij}\Gamma^k_{r\ell} X_{,k}$

$$X_{,i\ell} = \Gamma^k_{i\ell} X_{,k} + h_{i\ell} U$$

$$X_{,i\ell j} = \Gamma^k_{i\ell,j} X_{,k} + \Gamma^k_{i\ell} X_{,kj} + h_{i\ell,j} U + h_{i\ell} U_{,j}$$

$$\boxed{= \Gamma^k_{i\ell,j} X_{,k} + \Gamma^k_{i\ell}\left\{\Gamma^r_{kj} X_{,r} + \overset{*}{h_{kj}} U\right\} + h_{i\ell,j} U - h_{i\ell} a^k_j X_{,k}}$$

rewrite $\Gamma^r_{i\ell}\Gamma^k_{rj} X_{,k}$

$X_{,k}$ coefficient of $X_{,ij\ell} = X_{,i\ell j}$:

$$\Gamma^k_{ij,\ell} + \Gamma^r_{ij}\Gamma^k_{r\ell} - h_{ij} a^k_\ell = \Gamma^k_{i\ell,j} + \Gamma^r_{i\ell}\Gamma^k_{rj} - h_{i\ell} a^k_j$$

$$g^{si}\left\{ -h_{ij} a^k_\ell + h_{i\ell} a^k_j = \underbrace{(\Gamma^k_{i\ell,j} - \Gamma^k_{ij,\ell}) + (\Gamma^r_{i\ell}\Gamma^k_{rj} - \Gamma^r_{ij}}\Gamma^k_{r\ell}) \right.$$

$$-a^s_j a^k_\ell + a^s_\ell a^k_j = g^{si}(\qquad\qquad)$$

$j=k=1$, $s=\ell=2$: $$\boxed{K = \overset{*}{g^{2i}}\left(\Gamma^1_{i2,1} - \Gamma^1_{i1,2} + \Gamma^r_{i2}\Gamma^1_{r1} - \Gamma^r_{i1}\Gamma^1_{r2}\right)}$$

each reduces to book formula for orthog. param.

$j=k=2$, $s=\ell=1$: * $$\boxed{K = g^{1i}\left(\Gamma^2_{i1,2} - \Gamma^2_{i2,1} + \Gamma^r_{i1}\Gamma^2_{r2} - \Gamma^r_{i2}\Gamma^2_{r1}\right)}$$

Fun (hard) exercise (not to hand in).
Find general formula for Γ^k_{ij} for an arbitrary metric $[g_{rs}]$, not assumed orthogonal.

The Christoffel symbol formula in general

(without assuming orthogonal parameterization)

$$X_{,ij} = \Gamma_{ij}^{k} X_{,k} + h_{ij} U$$

$$\Rightarrow \quad X_{,ij} \cdot X_{,\ell} = \Gamma_{ij}^{k} g_{k\ell} + 0$$

$$\Rightarrow \quad g^{\ell r} (X_{,ij} \cdot X_{,\ell}) = \Gamma_{ij}^{k} \underbrace{g_{k\ell} g^{\ell r}}_{\delta_k^r}$$

$$g^{\ell r} (X_{,ij} \cdot X_{,\ell}) = \Gamma_{ij}^{r}$$

step 1 $\boxed{\Gamma_{ij}^{k} = g^{\ell k} (X_{,ij} \cdot X_{,\ell})}$

$$X_{,ij} \cdot X_{,\ell} = \underbrace{(X_{,i} \cdot X_{,\ell})_{,j}}_{g_{i\ell,j}} \underbrace{- X_{,i} \cdot X_{,\ell j}}_{\underbrace{- X_{,i} \cdot X_{,j\ell}}_{-[\, g_{ij,\ell} - X_{,i\ell} \cdot X_{,j} \,]}}$$

$$= g_{i\ell,j} - g_{ij,\ell} + \underbrace{X_{,\ell i} \cdot X_{,j}}_{g_{\ell j,i} - \underbrace{X_{,\ell} \cdot X_{,ji}}_{\text{started with this!}}}$$

$$\Rightarrow X_{,ij} \cdot X_{,\ell} = \tfrac{1}{2} (g_{i\ell,j} + g_{j\ell,i} - g_{ij,\ell})$$

$$\Rightarrow \boxed{\Gamma_{ij}^{k} = \tfrac{1}{2} g^{k\ell} (g_{i\ell,j} + g_{j\ell,i} - g_{ij,\ell})}$$

Exercise (last one for this Friday).

Verify this formula gives correct Christoffel symbols for 1st "acceleration" formula

$$X_{uu} = \Gamma_{uu}^{u} X_u + \Gamma_{uu}^{v} X_v + h_{11} U$$

$$= \text{(worked out before)}$$

$$= \frac{E_u}{2E} X_u - \frac{E_v}{2G} X_v + h_{11} U.$$

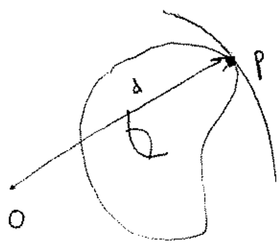
A surprising theorem, given all of our (noncompact) constant Gauss curvature surface examples. (See p 141 for an alternate proof.).

<u>Theorem</u>: Let M^2 be a regular compact surface in $\mathbb{R}^3$, of constant Gauss curvature K. Then $K > 0$ and M is a sphere of radius $R = \frac{1}{\sqrt{K}}$.

<u>Lemma</u> $K > 0$.

<u>proof</u> Since M is compact $\exists$ a point $p \in M$ s.t. $|p| \geq |q|$ $\forall q \in M$

i.e. p is maximal distance from origin



Let d be this distance.

We claim $\vec{U}(p) = \pm \frac{\vec{p}}{d}$

<u>pf</u> Let $\alpha : I \to M$

$\alpha(0) = p$

then $\alpha \cdot \alpha$ has a max at $t = 0$

$\Rightarrow \alpha'(0) \cdot \alpha(0) = 0$

$\Rightarrow \alpha(0) \perp \vec{v} \ \forall \vec{v} \in T_pM$.

Take $\vec{U}(p) = -\frac{\vec{p}}{d}$.

We claim each normal curvature $k_n(\vec{v}) \geq \frac{1}{d}$ (geometrically clear!).

<u>pf</u>: Choose α, $\alpha(0) = p$

$\alpha'(0) = \vec{v}$ (unit).

$$k_n(\vec{v}) = \alpha''(0) \cdot U$$
$$= \alpha''(0) \cdot \left(-\frac{\vec{\alpha}(0)}{d}\right).$$

But $\alpha \cdot \alpha$ has a local max @ $t = 0$

$\Rightarrow (\alpha \cdot \alpha)'(0) = 0$

$(\alpha \cdot \alpha)''(0) \leq 0$;

$(\alpha \cdot \alpha)' = 2\alpha \cdot \alpha'$

$(\alpha \cdot \alpha)'' = 2\alpha' \cdot \alpha' + 2\alpha \cdot \alpha'' \leq 0$

$\Rightarrow \alpha \cdot \alpha''(0) \leq -1$

$\Rightarrow k_n(\vec{v}) = -\frac{\alpha \cdot \alpha''(0)}{d} \geq \frac{1}{d}$.

$\Rightarrow k_1, k_2 \geq \frac{1}{d}$

$\Rightarrow K \geq \frac{1}{d^2} > 0$ ∎

proof of Theorem:

Choose $\vec{U}$ so that $k_1, k_2 > 0$ on all of M^2.

Case I M^2 is umbilic ($k_1 \equiv k_2$); in this case we proved before midterm that M^2 is a sphere of radius $R = \frac{1}{k_i}$ $\Rightarrow$ Theorem true

Case II M^2 is not umbilic
(we shall rule this case out. The proof is reminiscent of 2nd derivative test, is actually a special case of the maximum principle in elliptic PDE's).

Pick $p \in M$ s.t. $k_1(p)$ is the maximum principle curvature on M
(so also $k_2(p)$ is the minimum such.).

Thus $\forall\, q \in M$

$$\boxed{k_2(p)|w|^2 \le k_2(q)|w|^2 \le S_q(w)\cdot w \le k_1(q)|w|^2 \le k_1(p)|w|^2}$$

Now, Parameterize M near p, above T_pM

$$X(u,v) = p + u\vec{e}_1 + v\vec{e}_2 + f(u,v)\underbrace{U(p)}_{\vec{e}_3}$$

at p:
$S(\vec{e}_1) = k_1(p)\vec{e}_1$
$S(\vec{e}_2) = k_2(p)\vec{e}_2$.

$$[g_{ij}] = \begin{bmatrix} 1+f_u^2 & f_uf_v \\ f_vf_u & 1+f_v^2 \end{bmatrix}, \quad [h_{ij}] = \frac{1}{1+f_u^2+f_v^2}\begin{bmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{bmatrix}$$

Near $(u,v) = (0,0)$ deduce

$$k_1(p) \le \frac{h_{11}}{g_{11}} \qquad \text{equality at } (u,v) = (0,0)$$

$$\frac{h_{22}}{g_{22}} \le k_2(p) \qquad \text{equality at } (u,v) = (0,0)$$

So Calculus $\Rightarrow$
at $(u,v) = (0,0)$: $\left(\frac{h_{ii}}{g_{ii}}\right)_{,j} = 0 \Rightarrow h_{ii,j} = 0 \quad i,j = 1,2$, since $g_{ii,j} = 0$ there.

$$* \quad \left(\frac{h_{11}}{g_{11}}\right)_{,22} \ge 0, \quad \left(\frac{h_{22}}{g_{22}}\right)_{,11} \le 0.$$

But $\left(\frac{h_{11}}{g_{11}}\right)_{,22} = \left(\frac{h_{11,2}\,g_{11} - h_{11}\,g_{11,2}}{g_{11}^2}\right)_{,2} = h_{11,22}$ at $(0,0)$ check!

$\left(\frac{h_{22}}{g_{22}}\right)_{,11} = h_{22,11}$ at $(0,0)$

$* \Rightarrow h_{11,22} - h_{22,11} \ge 0$ at $(0,0)$. i.e. $\left(\frac{f_{uu}}{1+f_u^2+f_v^2}\right)_{vv} - \left(\frac{f_{vv}}{1+f_u^2+f_v^2}\right)_{uu} \ge 0$ at $(0,0)$

after canceling: $f_{uu}(-2f_{vv}^2) - f_{vv}(-2f_{uu}^2) \ge 0$ at $(0,0)$

$-2k_1k_2(k_2+k_1) \ge 0$ $\cancel{*}$ Since $K(p), k_1(p), k_2(p) > 0$

so case II didn't happen! ▨