Math 4530
Fri March 11

I'll post some additional HW over the break & send you all an email.

We're proving Gauss' Theorem Egregium.
See Wed notes for first go.

There's a more efficient way to do these computations

Warm-up.

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$ matrix of 1st fdl form

$$\begin{bmatrix} \ell & m \\ m & n \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$ matrix of 2nd fdl form

$g_{ij} = X_{,i} \cdot X_{,j}$
where $X = X(u^1, u^2)$
$X_{,i}$ means X_{u_i}

$h_{ij} = -\vec{U}_{,i} \cdot X_{,j} = X_{,ij} \cdot \vec{U}$

this notation is especially better in higher dimensions!

$$[S] = \begin{bmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{bmatrix}$$ almost as before.

$$S(X_{,j}) = \sum_k a^k_j X_{,k} = a^k_j X_{,k}$$

repeated roman letter indices are summed!

summation convention.

Re-derivation of $[S]_{\{X_u, X_v\}} = [I]^{-1}[II]$:

$$S(X_{,j}) = a^k_j X_{,k}$$

$$S(X_{,j}) \cdot X_{,\ell} = a^k_j X_{,k} \cdot X_{,\ell}$$

$$h_{j\ell} = a^k_j g_{k\ell}$$

$$[g_{ij}]^{-1} := [g^{ij}]$$

$$h_{j\ell} g^{\ell i} = a^k_j \underbrace{g_{k\ell} g^{\ell i}}_{\delta^i_k \ \left(= \begin{smallmatrix} 1 & \text{if } i=k \\ 0 & \text{if } i \neq k \end{smallmatrix}\right)}$$

mimics

$$\begin{bmatrix} S(X_u) \,\big|\, S(X_v) \end{bmatrix} = \begin{bmatrix} X_u \,\big|\, X_v \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$\vdots$

$$h_{j\ell} g^{\ell i} = a^i_j$$

$$\parallel$$

$$\boxed{g^{i\ell} h_{\ell j} = a^i_j}$$

because both matrices are symmetric.

Re-derivation of Gauss' Eqtn:

$$X_{,ij} = \Gamma^k_{ij} X_{,k} + h_{ij} U$$

$$X_{,ij\ell} - X_{,i\ell j} = 0 !$$

$$X_{,ij\ell} = \Gamma^k_{ij,\ell} X_{,k} + \Gamma^k_{ij} X_{,k\ell} + h_{ij,\ell} U + h_{ij} U_{,\ell}$$

$$\boxed{= \Gamma^k_{ij,\ell} X_{,k} + \Gamma^k_{ij} (\Gamma^r_{k\ell} X_{,r}) + h_{ij,\ell} U - h_{ij} a^k_\ell X_{,k}}$$

rewrite $\Gamma^r_{ij} \Gamma^k_{r\ell} X_{,k}$

$$X_{,i\ell} = \Gamma^k_{i\ell} X_{,k} + h_{i\ell} U$$

$$X_{,i\ell j} = \Gamma^k_{i\ell,j} X_{,k} + \Gamma^k_{i\ell} X_{,kj} + h_{i\ell,j} U + h_{i\ell} U_{,j}$$

$$\boxed{= \Gamma^k_{i\ell,j} X_{,k} + \Gamma^k_{i\ell} \Gamma^r_{kj} X_{,r} + h_{i\ell,j} U - h_{i\ell} a^k_j X_{,k}}$$

rewrite

$\Gamma^r_{i\ell} \Gamma^k_{rj} X_{,k}$

$X_{,k}$ coefficient of $X_{,ij\ell} = X_{,i\ell j}$:

$$\Gamma^k_{ij,\ell} + \Gamma^r_{ij} \Gamma^k_{r\ell} - h_{ij} a^k_\ell = \Gamma^k_{i\ell,j} + \Gamma^r_{i\ell} \Gamma^k_{rj} - h_{i\ell} a^k_j$$

$$g^{si} \Big\{ \quad -h_{ij} a^k_\ell + h_{i\ell} a^k_j = \underbrace{(\Gamma^k_{i\ell,j} - \Gamma^k_{ij,\ell}) + (\Gamma^r_{i\ell} \Gamma^k_{rj} - \Gamma^r_{ij} \Gamma^k_{r\ell})}$$

$$-a^s_j a^k_\ell + a^s_\ell a^k_j = g^{si} (\qquad\qquad)$$

$j=k=1$, $s=\ell=2$:

$$\boxed{K = g^{1i} (\Gamma^1_{i2,1} - \Gamma^1_{i1,2} + \Gamma^r_{i2} \Gamma^1_{r1} - \Gamma^r_{i1} \Gamma^1_{r2})}$$

reduces to book formula
for orthog. param.

Fun (hard) exercise (not to hand in).
Find general formula for Γ^k_{ij} for an arbitrary metric $[g_{rs}]$,
not assumed orthogonal.