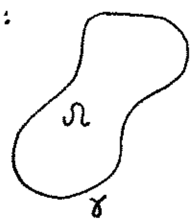
Math 4530
Fri 28 Jan

Go to LCB 115 on Monday
I'll post HW late today or Saturday!

Isoperimetric inequality: The answer to the question – what shape should you make a simple closed curve in the plane to maximize the enclosed area?

We have a fine experiment to motivate the answer!

<u>Theorem</u>:



Let γ be a simple closed curve in the plane, γ p.b.a.l., $\gamma : [0,L] \to \mathbb{R}^2$ the boundary of a region Ω with area A

Then $4\pi A \leq L^2$

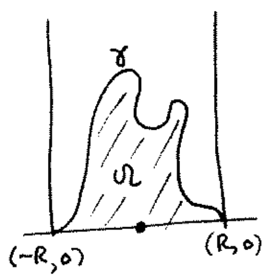
With equality only when γ is a circle (and Ω is a disk)

Notice: this says that for fixed length you get most area from a disk & that for fixed area the least length is the circle.

Also, the isoperimetric inequality is <u>scale</u> <u>invariant</u> (any such inequality must be!)

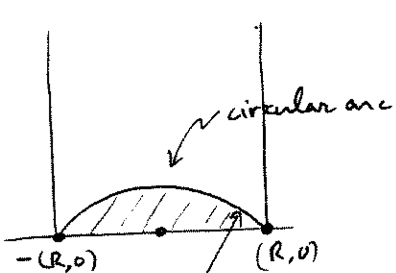
<u>proof</u> (From a paper by Gary Lawler, not exactly book's version)

<u>Key lemma</u> (will let us replace part of γ with circular arcs, decreasing length and preserving area).



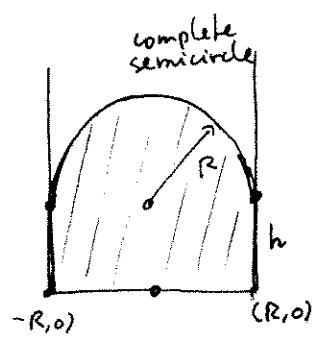
Let γ be a (simple) curve in the strip $[-R,R] \times [0,\infty]$, $\gamma(0) = [-R,0]$, $\gamma(L) = [R,0]$ so that $\gamma \cup [-R,R]$ bounds a region Ω with area A.

The γ is at least as long as the unique curve in the following list which also bounds a region of area A. Unless γ is this curve it is strictly longer



$0 < A \leq \frac{1}{2}\pi R^2$

$\sqrt{R}$



$A > \frac{1}{2}\pi R^2$

(note length $= 2h + \pi R$).

We could make a demo for this! (But that's not the proof!)

Key lemma proof :

Call our claimed length minimizing curve α

A clever use of the $n=2$ divergence theorem lets us compare the length of α to that of γ, using a specially constructed vector field.

(The text uses Green's thm, which is equiv. to div thm, although their proof is different in more respects.)

$$\iint_{\Omega} \operatorname{div}\vec{F}\, dA = \int_{\partial\Omega} \vec{F}\cdot \underset{\substack{\uparrow \\ \text{unit ext} \\ \text{normal.}}}{\vec{n}}\, ds$$

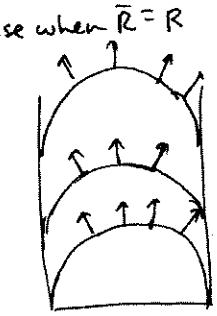
Let the arc of α have radius $\bar{R} \geq R$, and center on the y-axis

Let $\vec{F} := \frac{1}{\bar{R}}\langle x, \sqrt{\bar{R}^2 - x^2}\rangle$

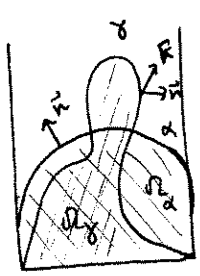
Virtues of $\vec{F}$

- $|\vec{F}| \equiv 1$
- $\vec{F} = \vec{n}$, the exterior normal, on α! check! $\rightarrow$ the arcs of α are level curves of $f(x,y) = y - \sqrt{\bar{R}^2 - x^2}$
  $\nabla f = \langle \frac{x}{\sqrt{\ }}, 1\rangle \parallel \vec{F}$, $\perp$ level curves
- $\operatorname{div}\vec{F} \equiv \text{const} \quad (= \frac{1}{\bar{R}})$

case when $\bar{R} = R$



Proof:



γ, α enclose regions $\Omega_\gamma, \Omega_\alpha$ of equal area

s $\operatorname{div}\vec{F} \equiv \text{const}$

$$\Rightarrow \iint_{\Omega_\alpha} \operatorname{div}\vec{F} = \iint_{\Omega_\gamma} \operatorname{div}\vec{F}$$

div thm

$$\Rightarrow \int_\alpha \underbrace{\vec{F}\cdot\vec{n}}_{1}\, ds + \cancel{\int_{[-R,R]} \vec{F}\cdot\vec{n}\, ds} = \int_\gamma \vec{F}\cdot\vec{n}\, ds + \cancel{\int_{[-R,R]} \vec{F}\cdot\vec{n}\, ds}$$

$$\Rightarrow \text{length}(\alpha) = \int_\gamma \underbrace{(\vec{F}\cdot\vec{n})}_{\substack{|\vec{F}||\vec{n}|\cos\theta \\ 1 \quad 1}}\, ds$$

$$\leq \int_\gamma 1\, ds = \text{length}(\gamma).$$

Furthermore, equality iff $\cos\theta \equiv 1$
$\Rightarrow \alpha$ is a level curve of f above!

∎

Proof of isoperimetric inequality.

Let γ, Ω be given.

We modify γ, Ω is a series of steps which increase $4\pi A - L^2$.

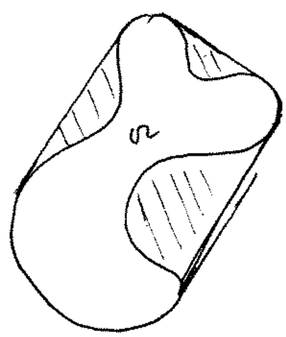
We end with a circle and a disk, for which $4\pi A - L^2 = 0$, and deduce that the original γ, Ω satisfied

$$4\pi A - L^2 < 0$$

(unless we started with ⊘)

step 1: Replace γ, Ω with the convex hull of $\Omega := \bigcup_{\vec{u}, \vec{v} \in \Omega} \{ t\vec{u} + (1-t)\vec{v},\ 0 \le t \le 1 \}$

This increases the area & decreases the length, so increases $4\pi A - L^2$, (strictly if Ω was not convex).



So now we may assume Ω is convex.

step 2 Since γ has finite length, Ω is bounded, $\bar{\Omega}$ is compact.

Find extreme points P, Q s.t.

$$|\vec{P} - \vec{Q}| = \max_{x, y \in \bar{\Omega}} d(x, y)$$

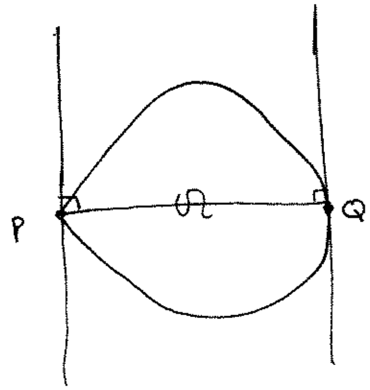
(d is cont. on $\bar{\Omega} \times \bar{\Omega}$ so attains extreme vals).

By maximality,

$$P, Q \in \partial\Omega = \gamma,$$

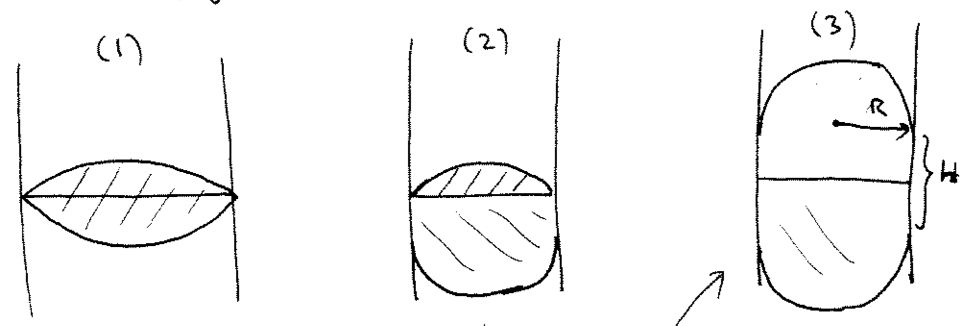
And $\bar{\Omega}$ lies in the strip with cross section $\overrightarrow{PQ}$:

(otherwise, maximality is contradicted.)

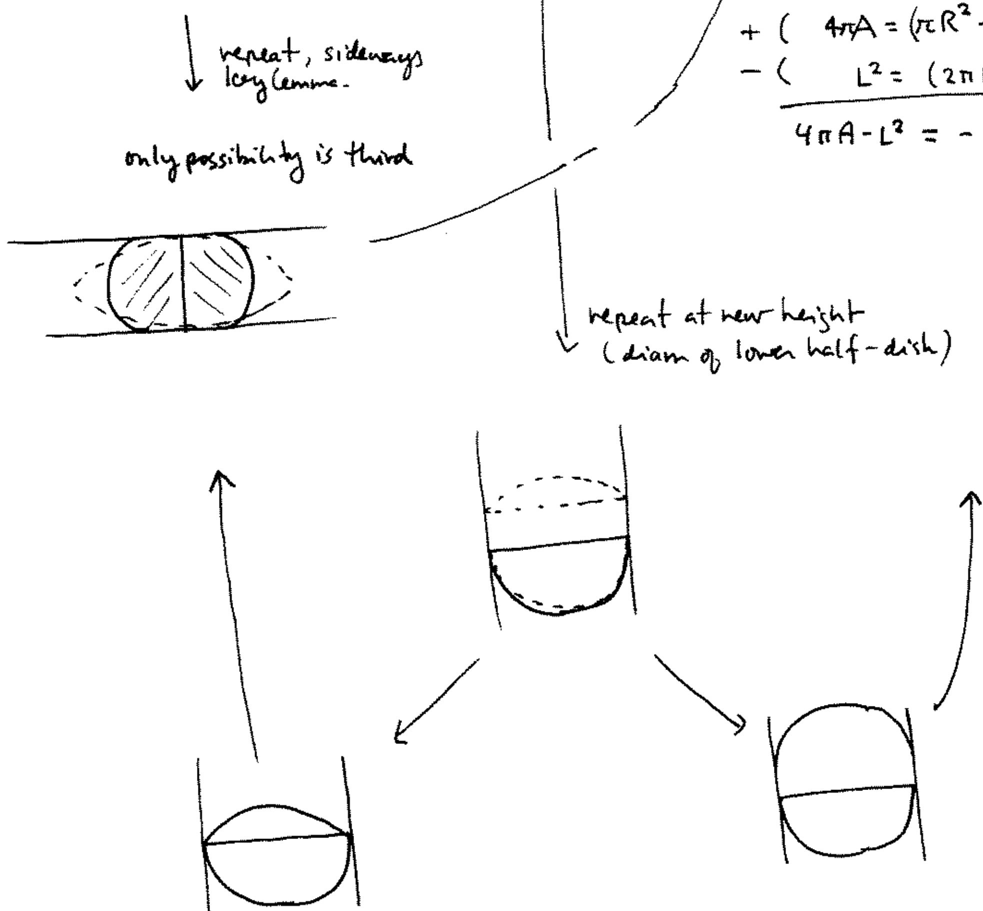


Now apply key lemma in half strips above & below $\overrightarrow{PQ}$.

Without changing A you decrease L and construct a new Ω which is one of:



$$+\left(\quad 4\pi A = (\pi R^2 + RH)\,4\pi\right.$$
$$-\left(\quad L^2 = (2\pi R + 2H)^2\right.$$
$$4\pi A - L^2 = -4H^2 < 0 \text{ unless } H = 0.$$



Always reduce to (3)!

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