Math 4530
Monday 1/24

Frenet Eqtns.
$\exists!$ thm for curves of presribed curvature & torsion

Recall, for α p.b.a.l. (∞'ly differentiable, parameterized by arclength)

$\boxed{\alpha'(s) := T(s)}$ unit tang vector

$\boxed{|T'(s)| := \kappa(s)}$ curvature

→ if $|T'| \equiv 0$ on (sub)interval, $T \equiv T_0$
$\alpha \equiv s\vec{T}_0 + \vec{\alpha}_0$, a line

→ assume $|T'| > 0$ on (sub) interval

$\vec{N}: \frac{T'}{|T'|} = \frac{1}{\kappa} T'$

so,

$\boxed{T' = \kappa N}$

the $\{T, N\}$ plane thru $\alpha(s)$ is called the <u>osculating plane</u>, it is the only plane containing $\alpha(s+\Delta s)$ through 2nd order, at $\alpha(s)$: $(\kappa \neq 0)$

Taylor: $\alpha(s+\Delta s) = \alpha(s) + \Delta s \underbrace{\alpha'(s)}_{T} + \frac{1}{2}(\Delta s)^2 \underbrace{\alpha''(s)}_{\kappa N} + \underbrace{O(\Delta s)^3}$

big "O" of $(\Delta s)^3$ means its quotient by $(\Delta s)^3$ stays bounded as $\Delta s \to 0$

example:
osculating plane on a helix.

$\alpha(s) = \langle a\cos\frac{s}{c}, a\sin\frac{s}{c}, bs \rangle$ $c = \sqrt{a^2+b^2}$

$\alpha' = T = \langle -\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, b \rangle$

$T' =$

so $\kappa =$

$N =$

continuing

$T \times N := B$ the binormal (i.e. the normal vector of the osculating plane)

B' measures how fast this plane is changing

$$B \cdot B \equiv 1 \Rightarrow B' \cdot B \equiv 0$$

$$B \cdot T \equiv 0 \Rightarrow B' \cdot T + \underbrace{B \cdot \underbrace{T'}_{KN}}_{0} \equiv 0 \Rightarrow B' \cdot T \equiv 0$$

$$\Rightarrow B' \parallel N$$

write

$B' = -\tau N$ (i.e. $\tau := -B' \cdot N$)

$\tau :=$ torsion

example: In HW you compute τ for a-b helix, $\tau = \frac{b}{c^2}$

Frenet system : If $\alpha(s)$ is p.b.a.l. and has curvature $\kappa(s)$, torsion $\tau(s)$ then α, T, N, B satisfy a 1st order system of linear DE's

$$\left.\begin{aligned} \alpha' &= T \\ T' &= \kappa N \\ N' &= -\kappa T + \tau B \\ B' &= -\tau N \end{aligned}\right\} \text{Frenet system.}$$

pf: verify N' formula using dot product :

Corollary

Let $\kappa(s)$ (>0), $\tau(s)$, be defined C^∞ fcns on an interval $[0,L]$.
Then $\exists$ α p.b.a.l. with that curvature and torsion

α is unique up to a rigid motion of $\mathbb{R}^3$, i.e. up to choice of initial point $\alpha(0)$, and initial frame $T(0), N(0), B(0)$.

proof:

$\exists!$ theorem for linear systems of 1^{st} order DE's implies $\exists!$ sol'n to

$$\text{IVP}\quad\begin{cases}\alpha' = T\\ T' = \kappa N\\ N' = -\kappa T + \tau B\\ B' = -\tau N\\ \alpha(0) = \alpha_0\\ T(0) = T_0\\ N(0) = N_0\\ B(0) = B_0\end{cases}$$

(local existence for any 1^{st} order system, sol'n exists on entire interval if system is linear.)

Subtle point 1: if $\{T_0, N_0, B_0\}$ are positively oriented orthonormal frame, how do you know that $\{T, N, B\}$ also are. $\forall s \in [0,L]$
(You must know this in order to deduce that κ, τ are the curve curvature and torsion.)

Reason: If $\{T_0, N_0, B_0\}$ are initially an orthonormal frame then consider the system satisfied by

$x_1 := T\cdot T$ $\qquad x_1' = 2T'\cdot T = 2\kappa N\cdot T = 2\kappa x_2$

$x_2 := T\cdot N$ $\qquad x_2' = T'\cdot N + T\cdot N' = \kappa x_4 - \kappa x_2 - \kappa x_1 + \tau x_3$

$x_3 := T\cdot B$ $\qquad x_3' = T'\cdot B + T\cdot B' = \kappa x_5 - \tau x_2$

$x_4 := N\cdot N$ $\qquad x_4' = 2N'\cdot N = -2\kappa x_2 + 2\tau x_5$

$x_5 = N\cdot B$ $\qquad x_5' = N'\cdot B + N\cdot B' = -\kappa x_3 + \tau x_6 - \tau x_4$

$x_6 = B\cdot B$ $\qquad x_6' = 2B'\cdot B = -2\tau x_5$

$$\text{IVP}\quad\begin{cases}\begin{bmatrix}x_1'\\x_2'\\x_3'\\x_4'\\x_5'\\x_6'\end{bmatrix} = \begin{bmatrix}0 & 2\kappa & 0 & 0 & 0 & 0\\ -\kappa & -\kappa & \tau & \kappa & 0 & 0\\ 0 & -\tau & 0 & 0 & \kappa & 0\\ 0 & -2\kappa & 0 & 0 & 2\tau & 0\\ 0 & 0 & 0 & 0 & -2\tau & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}\\[2ex] \begin{bmatrix}x_1(0)\\x_2(0)\\x_3(0)\\x_4(0)\\x_5(0)\\x_6(0)\end{bmatrix} = \begin{bmatrix}1\\0\\0\\1\\0\\1\end{bmatrix}\end{cases}$$

but $\begin{bmatrix}1\\0\\0\\1\\0\\1\end{bmatrix}$ is a soltn. $\exists! \Rightarrow$ the sol'n!

subtle point 2 : Uniqueness up to rigid motion.

Let $\kappa(s), \tau(s)$ fixed on $[0,L]$.

Let α solve

$$\begin{cases} \alpha' = T \\ T' = \kappa N \\ N' = -\kappa T + \tau B \\ B' = -\tau N \\ \alpha(0) = \alpha_0 \\ T(0) = T_0 \\ N(0) = N_0 \\ B(0) = B_0 \end{cases}$$

$\{T_0, N_0, B_0\}$ orthonormal positive frame. $(\det[T_0|N_0|B_0] > 0)$

Let β solve

$$\begin{cases} \beta' = T \\ T' = \kappa N \\ N' = -\kappa T + \tau B \\ B' = -\tau N \\ \beta(0) = \beta_1 \\ T(0) = T_1 \\ N(0) = N_1 \\ B(0) = B_1 \end{cases}$$

$\{T_1, N_1, B_1\}$ orthonormal positive frame.

Consider the rigid motion (composition of translation & rotation) which transforms α's initial data into β's:

$$\rho(x) := \underbrace{x + (\beta_1 - \alpha_0)}_{\text{translation}} + \underbrace{\left[T_1 | N_1 | B_1\right]\left[T_0 | N_0 | B_0\right]^{-1}}_{\text{rotation matrix } \theta} \vec{x}$$

HW I for Friday

Prove $\beta(s) = \rho(\alpha(s))$