Math 4530-1
Fri 21 Jan

(Notes modified from originals of Hugo Rossi)

Kepler's Laws and Newton's mathematical model:
the origins of the inverse square law of gravitational attraction

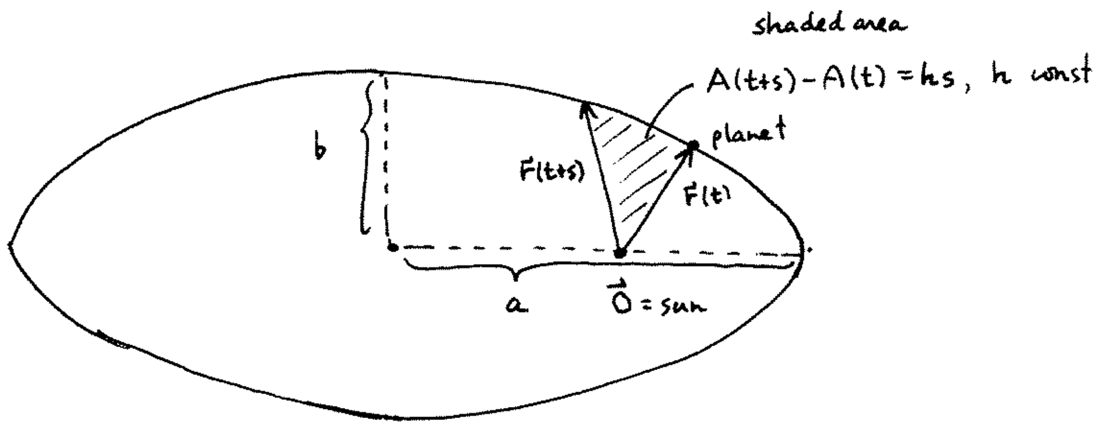
descriptive, based on painstaking observations of planetary motion:

KEPLER

(I) The orbit of a planet (a curve with position vector $\vec{r}(t)$, t = time) from $\vec{O}$ = sun location
is an ellipse lying in a fixed plane. The sun is at a <u>focus</u> of the ellipse.

(II) Equal areas are swept out in equal times; if $A(t)$ is area swept out by $\vec{r}(t)$ (after some initial time t_0), then $A(t+s) - A(t) = hs$, h constant.
i.e. $A'(t) \equiv h$, constant. (Kepler didn't know calculus, though, that was another discovery of Newton.)

(III) Let T = period of orbit. Let a = ½-length of major axis for ellipse
Then $T = \tilde{c}\, a^{3/2}$ $(T^2 \sim a^3)$, with the constant $\tilde{c}$ <u>independent</u> of the planet
(but it may depend on the "sun")



Newton derived a model to explain Kepler, and this is also the origin of <u>his discovery</u> of the inverse square law of gravitational attraction:

$$|F| = \frac{GmM}{r^2}$$

G = univ constant
m, M = object masses
$r = |\vec{r}|$ = separation distance

NEWTON

Newton:

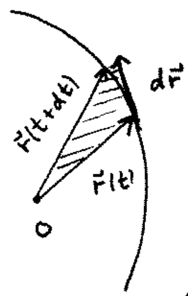
① An orbiting planet orbits in a fixed plane containing the sun, and satifies the equal area condition (II) if and only if the acceleration is <u>centripetal</u> $(\parallel \vec{r})$

② The only centripetal acceleration consistent with (I) is one with magnitude $\frac{c}{r^2}$ (inverse square law), c = constant, for a given planet

③ (III) holds if and only if the constant c in ② is the same for each planet's orbit
(= GM, as it turns out.)

This discussion incorporates almost all of the ideas we've talked about so far (except p.b.a.l.). (In addition to being a historical breakthrough in physics & math.)

Lemma 1a If the orbit lies in a plane containg the sun and if equal area II holds, then the acceleration is centripetal:



in time dt,

$dA = \frac{1}{2}|\vec{r} \times d\vec{r}|$ (triangle area) geometric meaning of cross product!

$= \frac{1}{2}|\vec{r} \times \vec{r}'(t)dt|$ and definition of derivative

so

(1.1) $$\boxed{\frac{dA}{dt} = \frac{1}{2}|\vec{r} \times \vec{r}'|}$$

If $\vec{r}$, hence $\vec{r}'$ lie in plane thru sun, then $\vec{r}\times\vec{r}'$ is a multiple of this plane's unit normal vector "$\vec{k}$". (since $\vec{r}\times\vec{r}' \perp \vec{r}$ and $\vec{r}'$)

If $\frac{dA}{dt} \equiv h$ const (II),

the multiple of $\vec{k}$ which $\vec{r}\times\vec{r}'$ is, is constant, i.e.

(1.2) $$\boxed{\vec{r}\times\vec{r}' = 2h\vec{k}}$$ (if we orient $\vec{k}$ as usual)

Thus

(1.3) $$\boxed{\frac{d}{dt}(\vec{r}\times\vec{r}') \equiv \vec{0}}$$

$$\parallel$$

$$\underbrace{\vec{r}'\times\vec{r}'}_{=0} + \vec{r}\times\vec{r}''$$

so

(1.4) $$\boxed{\vec{r}\times\vec{r}'' \equiv \vec{0}}$$ i.e. $\vec{r}'' \parallel \vec{r}$, centripetal accel. ∎

Lemma 1b If the acceleration is centripetal, then the planet orbits the sun in a fixed plane containing the sun, and the equal area condition II holds

proof: if $\vec{r}'' \parallel \vec{r}$ then (1.4) holds, so also

(1.3): $\frac{d}{dt}(\vec{r}\times\vec{r}') \equiv \vec{0}$

antidifferentiating yields $\vec{r}\times\vec{r}' = \vec{K}$, $\vec{K}$ constant

in particular $\vec{r}\cdot\vec{K} \equiv 0$

so $\vec{r}(t)$ lies in the plane $\langle x,y,z\rangle \cdot \vec{K} = 0$, which contains the origin (= sun).

choose coords so that the plane is the x-y plane, and $\vec{K} = 2h\hat{k}$ $h>0$ const

Thus (1.2) holds, and (1.1) implies $\frac{dA}{dt} \equiv h$ ∎

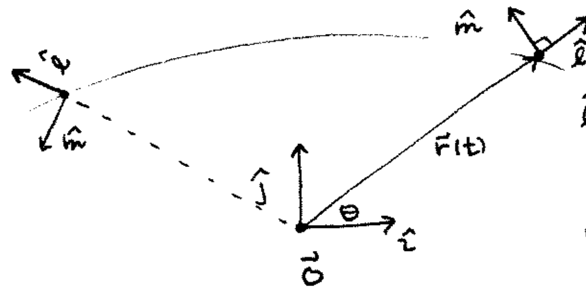
To prove Newton ② ③ we introduce a new "frame" (moving basis)

old, fixed frame:

plane is x-y plane, basis $\hat{\imath}, \hat{\jmath}$

normal vector $\hat{k}$

planet orbits counterclockwise so $\vec{r} \times \vec{r}\,'$ is positive multiple of $\hat{k}$.



$\hat{\ell} = \dfrac{\vec{r}}{r}$ $\quad (r = |\vec{r}|)$

$\hat{\ell} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$

$\hat{m} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$

$\hat{\ell} \times \hat{m} = \hat{k}$.

moving frame along $\vec{r}(t)$ given by $(\hat{\ell}, \hat{m}, \hat{k})$

Using r and θ to describe particle location we notice

<u>Lemma 2</u>

(i) $\dfrac{dA}{dt} = \dfrac{r^2}{2}\dfrac{d\theta}{dt}$ $\qquad \left(dA = \dfrac{r^2}{2}\, d\theta \right)$

$\dfrac{dA}{dt} = h$, so $\boxed{\theta' = \dfrac{2h}{r^2}}$

(ii) $\vec{r} = r\hat{\ell}$, so $\vec{r}\,' = r'\hat{\ell} + r\hat{\ell}'$

(iii) $\hat{\ell}' = \theta'\hat{m}$

(iv) $\hat{m}' = -\theta'\hat{\ell}$

Proof of Newton ② ③ :

centripetal acceleration

$$\Rightarrow \quad \vec{r}\,'' = -f\,\hat{\ell}$$, where a priori f could depend on anything, e.g. time, $\vec{r}$, ...

$$\Rightarrow \quad \vec{r}\,'' \times \hat{k} = -f\,\hat{\ell} \times \hat{k}$$
$$= f\,\hat{m}$$
$$= f\,\frac{1}{\theta'}\,\hat{\ell}\,' \qquad \text{lemma 2.iii}$$

(3.1a) $$\boxed{\vec{r}\,'' \times \hat{k} = f\,\frac{r^2}{2h}\,\hat{\ell}\,'} \qquad \text{lemma 2i}$$

If $f = f(r) = \frac{c}{r^2}$ [this is for half of Newton's thm]

(3.1b) $$\Rightarrow \boxed{\vec{r}\,'' \times \vec{k} = \frac{c}{2h}\,\hat{\ell}\,'}$$ [the r^2's cancel]

$$\Rightarrow \quad \vec{r}\,' \times \vec{k} = \frac{c}{2h}\,\hat{\ell} + \vec{E}$$ $\vec{E}$ is constant of integration, is $\perp \vec{k}$ since all the other terms are.

(3.2) $$\boxed{(r'\hat{\ell} + r\underbrace{\hat{\ell}\,'}_{\theta'\hat{m}}) \times \vec{k} = \frac{c}{2h}\,\hat{\ell} + \vec{E}}$$

for later, check $\hat{m}$ components too:
$$-r' = \vec{E} \cdot \hat{m}$$

dot both sides with $\hat{\ell}$:

$$r\theta' = \frac{c}{2h} + \vec{E} \cdot \hat{\ell}$$

lemma 2.i $$\frac{2h}{r} = \frac{c}{2h} + \vec{E} \cdot \hat{\ell}$$

$$\frac{r}{2h} = \frac{1}{\frac{c}{2h} + \vec{E} \cdot \hat{\ell}}$$

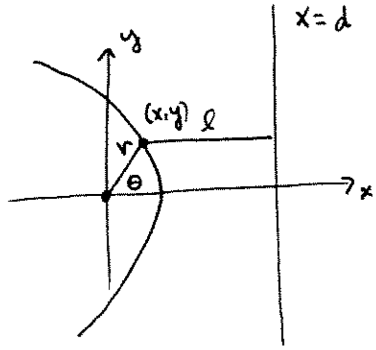
(3.3) $$\boxed{r = \frac{4h^2/c}{1 + \frac{2h}{c}\,\vec{E} \cdot \langle \cos\theta, \sin\theta \rangle}}$$

this is a polar coord eqn for something!

Notice, we can choose our original $\hat{\imath}, \hat{\jmath}$ so that $\vec{E} = \lambda\hat{\imath}$, so

(3.4) $$\boxed{r = \frac{4h^2/c}{1 + \frac{2h\lambda}{c}\cos\theta}}$$

Conic sheet:



$\frac{r}{\ell} = e$ constant, called eccentricity

$$\frac{r}{d - r\cos\theta} = e$$

$$r = e(d - r\cos\theta)$$

$$r(1 + e\cos\theta) = ed$$

$$\boxed{r = \frac{ed}{1 + e\cos\theta}}$$

$e < 1$ ellipse
($e = 0$ circle)
$e = 1$ parabola
$e > 1$ hyperbola

so the 2nd page 4 box says we have a conic, with

$$\boxed{e = \frac{2h\lambda}{c}, \quad d = \frac{2h}{\lambda}}$$

(we only call them "planets" when this conic is an ellipse!)

To explain Kepler III ($T^2 \sim a^3$), derive cartesian eqtn of ellipse:

$$\frac{r}{\ell} = e$$

$$\sqrt{x^2+y^2} = e(d-x)$$

$$x^2+y^2 = e^2(x^2 - 2dx + d^2)$$

$$x^2(1-e^2) + 2de^2x \quad + y^2 = e^2d^2$$

$$(1-e^2)\left[x^2 + \frac{2de^2}{1-e^2}x\right] + y^2 = e^2d^2$$

$$(1-e^2)\left[x + \frac{de^2}{1-e^2}\right]^2 + y^2 = e^2d^2 + \frac{d^2e^4}{1-e^2}$$

$$= \frac{e^2d^2}{1-e^2}$$

$$\frac{\left(x + \frac{de^2}{1-e^2}\right)^2}{\underset{\uparrow \atop a^2}{\left(\frac{ed}{1-e^2}\right)^2}} + \frac{y^2}{\underset{\uparrow \atop b^2}{\left(\frac{ed}{\sqrt{1-e^2}}\right)^2}} = 1$$

so $b = \sqrt{a}\sqrt{ed} = \sqrt{a}\,\frac{2h}{\sqrt{c}}$

Why is this ellipse area?
(Think about area scaling when you stretch plane)

now, $\frac{dA}{dt} = h$, so $\int_0^T h\,dt$ = ellipse area $= \pi ab$

$$\| $$

$$hT = \pi ab$$

$$T = \frac{\pi ab}{h} = \frac{\pi a \sqrt{a}\,\frac{2\cancel{h}}{\sqrt{c}}}{\cancel{h}} = \frac{\pi}{\sqrt{c}}\, a^{3/2}$$

Kepler III

c came from $f(r) = \frac{c}{r^2}$

By Kepler III, c doesn't depend on planet.

We've shown $f(r) = \frac{c}{r^2}$ (necessarily same c for each planet)

implies all of Kepler's laws

Can any other centripetal acceleration imply these laws?

Assuming Kepler got us the results on pages 2-3, through

$$(3.1a)\quad \vec{r}\,'' \times \hat{k} = f \frac{r^2}{2h} \hat{\ell}\,'$$

If we can work backwards from Kepler on page 6:

$$\boxed{r = \frac{ed}{1 + e\cos\theta}, \quad \frac{dA}{dt} = h}$$

to get (3.1b):

$$(3.1b)\quad \vec{r}\,'' \times \vec{k} = \frac{c}{2h} \hat{\ell}\,'$$

then deduce $f = c/r^2$.

So, $\lambda = \frac{2h}{d}$, $c = \frac{2h\lambda}{e}$ for box on this page, yields box on page 5 for e & d,

so get

(3.4) → (3.3) → (3.2) at least for the $\hat{\ell}$ component

to check the $\hat{m}$ comp. of (3.2), check

$$-r' = \hat{E} \cdot \hat{m} = \lambda \hat{\imath} \cdot \hat{m} = -\lambda \sin\theta \,?$$

$$r' = \lambda \sin\theta \,?$$

well, $r = \frac{ed}{1 + e\cos\theta}$

$$r' = -\frac{ed}{(1 + e\cos\theta)^2} (-e\sin\theta)\theta'$$

$$r' = \frac{r^2}{d} \sin\theta \underbrace{\theta'}_{\frac{2h}{r^2}}$$

$$r' = \frac{2h}{d} \sin\theta = \lambda \sin\theta \quad ✓$$

Thus (3.2), → (3.1b)! ∎