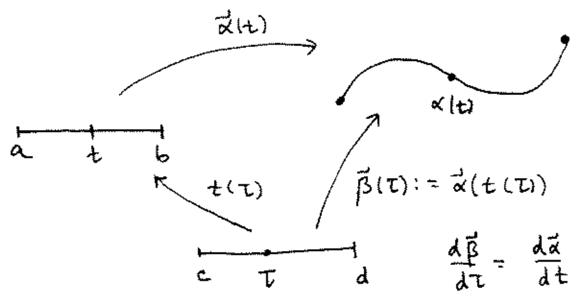
Math 4530
Wed 1/19

remember optional problem session tomorrow (Thurs.) for HW due Fri; LCB 322, 10-11:30

last week:
curves
$\alpha'(t), \alpha''(t)$
product rules for differentiating
dot product
cross product

We will also need the chain rule for reparameterizing curves:



$$\frac{d\vec{\beta}}{d\tau} = \underset{\text{vector}}{\frac{d\vec{\alpha}}{dt}} \; \underset{\text{scalar}}{\frac{dt}{d\tau}}$$

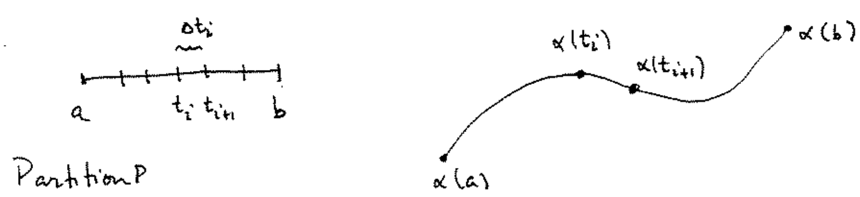
(what follows repeats part of last Wed's notes.)

Curve length

$\alpha: I = [a,b] \to \mathbb{R}^3$ differentiable (means continuously differentiable, i.e. $\alpha'(t)$ continuous)

Def1 length $L(\alpha) := \int_a^b \underbrace{|\alpha'(t)|}_{\text{speed}}\,dt$

Def2 $L(\alpha) = \lim_{\|P\| \to 0} \sum_i |\alpha(t_{i+1}) - \alpha(t_i)| = \sup_P \sum_i |\alpha(t_{i+1}) - \alpha(t_i)|$



Theorems: (A) Def2 defines length even for a continuous curve (may be $+\infty$ if curve "non-rectifiable")

idea: If you refine partition the approximate length increases, because of Δ inequality.
it is a good exercise in analysis to complete the proof.

(B) For diffble curves, Def1 & Def2 give same value of length. (You might have done this in analysis or Calculus)

(C) Def1 yields the same value for length regardless of how you reparameterize α $\to$ this is one of your HW problems; use chain rule

<u>Theorem</u> (you thought you knew this. Later we will study "shortest" curves on surfaces, called geodesics.)

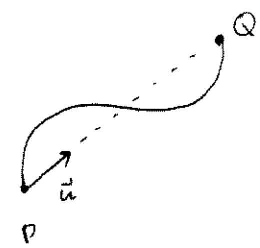
Let $P, Q \in \mathbb{R}^3$. Then the shortest path from P to Q is a straight line segment

<u>proof 1</u>: The straight-line distance is $|\vec{PQ}|$, as you
(using Def 1) can see immediately using

$$\vec{\beta}(t) = \vec{P} + t(\vec{Q} - \vec{P}) \quad , \quad 0 \le t \le 1.$$

Now let

$$\alpha : [a,b] \longrightarrow \mathbb{R}^3 \quad \text{diff'ble}, \quad \begin{array}{l} \alpha(a) = P \\ \alpha(b) = Q \end{array}$$



$$L = \int_a^b |\alpha'(t)|\, dt$$

$$\geq \int_a^b \alpha'(t) \cdot \vec{u} \;\; dt \qquad \vec{u} = \frac{\vec{PQ}}{|\vec{PQ}|} \text{ unit direction vector from P to Q}$$

notice inequality is strict
unless $\alpha'(t) = c(t)\,\vec{u} \quad \forall t, \; c(t) \geq 0$

$$\Rightarrow \alpha(r) = \alpha(a) + \int_a^r \underbrace{\alpha'(t)}_{c(t)\vec{u}}\, dt$$

$$= \alpha(a) + \left(\int_a^r c(t)dt\right)\vec{u}$$

$\Rightarrow \alpha$ parameterizes line segment from P to Q.

Notice $\alpha(t) \cdot \vec{u}$ is an antiderivative of the integrand, so

$$\int_a^b \alpha'(t) \cdot \vec{u}\, dt = \alpha(t) \cdot \vec{u} \Big]_a^b$$

$$= (\alpha(b) \cdot \vec{u}) - \alpha(a) \cdot u$$

$$= (Q - P) \cdot \frac{Q-P}{|Q-P|}$$

$$= |Q - P|.$$

So $L \geq |Q-P|$, with equality if and only if α parameterizes the segment from P to Q ∎

Let α as above.
<u>proof 2</u>: Consider the partition of $[a,b]$ consisting of $[a,b]$ itself.
(using Def 2) The approximate length is $|\vec{PQ}|$, which is the straight-line ~~d~~ length.
As you refine the partition, approximate lengths will strictly increase (by the strict form of Δ ineq.), unless $\alpha(t)$ moves monotonically from P to Q, along the line segment.

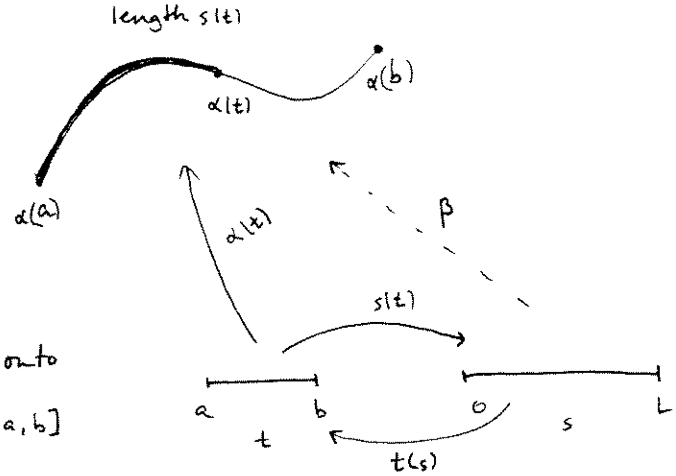
∎

parameterization by arclength

Let $\alpha : [a,b] \to \mathbb{R}^3$ differentiable, with $|\alpha'(t)| > 0 \ \forall t$ ("regular").

Define

$$s(t) := \int_a^t |\alpha'(\tau)|\,d\tau \quad = \text{length of curve arc between } \alpha(a) \ \& \ \alpha(t)$$



$s(a) = 0$

$s(b) = L$, the total length

$s'(t) = |\alpha'(t)| > 0$ so $s(t)$ is strictly increasing

Thus $s : [a,b] \to [0,L]$ is 1-1 & onto

so has inverse function $t : [0,L] \to [a,b]$

Define $\beta(s) := \alpha(t(s))$

Then $\beta'(s) = \alpha'(t)\,\frac{dt}{ds}$ (Chain rule)

$$|\beta'(s)| = |\alpha'(t)|\,\frac{1}{\frac{ds}{dt}} = \frac{|\alpha'(t)|}{|\alpha'(t)|} = 1$$

<u>so</u> $|\beta'(s)| = 1$ "unit speed parameterization"

& $\int_0^s \underbrace{|\beta'(r)|}_{1}\,dr = s$ "equal length parameterization"

$\Bigg\}$ $\boxed{\beta \text{ is parameterized by arclength}}$

<u>Example</u> (p. 18 text). Helix $\alpha(t) = (a\cos t, a\sin t, bt)$ ← circling a cylinder of radius a, with vertical speed b.

set $a = 0$. Find $s(t)$.

Reparameterize by arclength.

Note this includes circles ($b = 0$).

We study the geometry of curves by using arclength parameterizations.

Let $\beta : [0, L] \to \mathbb{R}^3$ be p.b.a.l. (parameterized by arclength)
and C^∞ (all derivs exist & are continuous)

$\beta'(s) := \vec{T}(s)$ unit tangent vector

$|T'(s)| := \kappa(s)$ curvature. Measures how fast $\vec{T}(s)$ is changing (wrt arclength), i.e. how fast curve is bending.

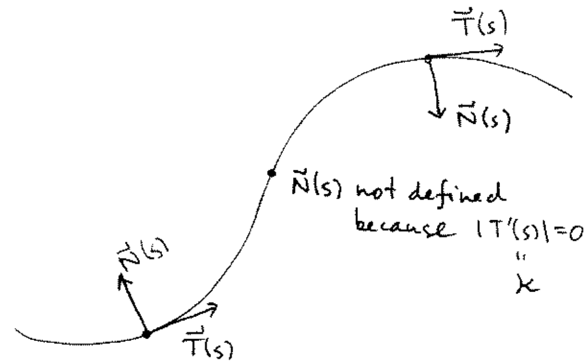
Note $\vec{T} \cdot \vec{T} \equiv 1$

so $2T'(s) \cdot \vec{T}(s) \equiv 0$

so $T'(s) \perp T(s)$

Write $\vec{N}(s) := \dfrac{T'(s)}{|T'(s)|}$ unit normal (when $\kappa \neq 0$)

So $\vec{T}'(s) = \kappa \vec{N}$



Formula from Calculus & Physics (mentioned last week).

Let $\vec{\alpha}(t)$ diffble curve
$\vec{\beta}(s)$ reparameterization by arclength

Then

$$\alpha''(t) = v'(t)\vec{T} + \kappa v^2 \vec{N} \qquad (v(t) = |\alpha'(t)| = \text{speed})$$

proof: $\vec{T} = \dfrac{\alpha'(t)}{|\alpha'(t)|}$

So $\alpha'(t) = v(t)\vec{T}(t)$

So $$\alpha''(t) = v'T + v \underbrace{\frac{d\vec{T}}{dt}}_{\underbrace{\frac{d\vec{T}}{ds}}_{\kappa\vec{N}} \underbrace{\frac{ds}{dt}}_{v}}$$

∎