Math 4530
Fri 1/14

Add ~~one~~ three more problems to 1/21 HW: (completes assignment)
1.1.24, 1.1.25 (p.16)
and 1.3.5 p.21

Recall the $\boxed{\text{universal product rule}}$ for suitable products of vector objects.
We indicate the hypotheses as they are needed in the proof. If your product satisfies the hypotheses, you deduce the product rule!

$$\boxed{\frac{d}{dt} A(t) * B(t) = A' * B + A * B'}$$

pf

$$\frac{d}{dt} A*B = \lim_{\Delta t \to 0} \frac{A(t+\Delta t) * B(t+\Delta t) - A(t) * B(t)}{\Delta t}$$

← hyp1 i.e. product yields a vector (or scalar) function.

$$= \lim_{\Delta t \to 0} \frac{A(t+\Delta t) * B(t+\Delta t) - A(t) * B(t+\Delta t)}{\Delta t} + \frac{A(t) * B(t+\Delta t) - A(t) * B(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left(\frac{A(t+\Delta t) - A(t)}{\Delta t}\right) * B(t+\Delta t) + A(t) * \left(\frac{B(t+\Delta t) - B(t)}{\Delta t}\right)$$

uses hyp1 (to move scalar Δt around)
and hyp2 $A*(B+C) = A*B + A*C$
$(A+B)*C = A*C + B*C$
(multiplication distributes over +).

$$= \left(\lim_{\Delta t \to 0} \frac{A(t+\Delta t) - A(t)}{\Delta t}\right) * \lim_{\Delta t \to 0} B(t+\Delta t)$$

$$+ A(t) * \left(\lim_{\Delta t \to 0} \frac{B(t+\Delta t) - B(t)}{\Delta t}\right)$$

hyp3 $*: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ is continuous

$$= A'(t) * B(t) + A(t) * B'(t)$$

uses lemma from hyp1
differentiable ⟹ continuous.

<u>examples</u>

$f(t)\vec{\alpha}(t)$ scalar · vector
$\vec{\alpha}(t) \cdot \vec{\beta}(t)$ dot prod
$\vec{\alpha}(t) \times \vec{\beta}(t)$ cross prod
$A(t)\vec{\alpha}(t)$ matrix times vect
$A(t)\, B(t)$ matrix times matrix
$\cancel{f \circ g(t)}$ composition
fails hyp2, which is why the chain rule does not look like the product rule!

More preliminaries

dot product

$$\vec{u} = \langle u_1, u_2, u_3 \rangle$$
$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$
$$\vec{u}\cdot\vec{v} = \sum_{i=1}^{3} u_i v_i$$

1 $\langle 1, 2, 3\rangle \cdot \langle 5, 2, -3\rangle =$

dot product algebra

(1) $\vec{u}\cdot\vec{u} = |\vec{u}|^2$

(2) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

(3) $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$

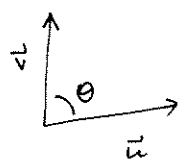
$(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$

(4) $(c\vec{u})\cdot\vec{v} = c(\vec{u}\cdot\vec{v}) = \vec{u}\cdot(c\vec{v})$

so

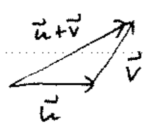
(5) $|c\vec{u}| = |c||\vec{u}|$; $\dfrac{\vec{v}}{|\vec{v}|}$ is always a unit vector.

Theorem $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$ See HW I(d) and text



Cor 1 $|\vec{u}\cdot\vec{v}| \le |\vec{u}||\vec{v}|$ (Cauchy Schwarz ineq.)

Cor 2 $|\vec{u}+\vec{v}| \le |\vec{u}| + |\vec{v}|$ (See HW I(e)) (Triangle inequality)



2 What is angle between vectors in 1 ?

More review

cross product

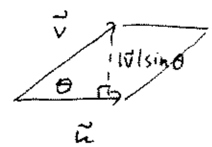
$\vec{u} = \langle u_1, u_2, u_3 \rangle$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \langle u_2 v_3 - v_2 u_3, \; -u_1 v_3 + u_3 v_1, \; u_1 v_2 - v_1 u_2 \rangle$$

Theorem

(i) $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$ = ▱ area :



(ii) $\vec{u} \times \vec{v} \perp$ to $\vec{u}, \vec{v}$,

and $\{\vec{u}, \vec{v}, \vec{u} \times \vec{v}\}$ are "right-handed" (positively oriented)

cross product algebra

(1) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

(2) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

$\}$ from det properties

(3) $\vec{w} \cdot (\vec{u} \times \vec{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ ($\Rightarrow$ Theorem (ii) $\vec{u} \times \vec{v} \perp \vec{u}, \vec{v}$).

(4) $|\vec{v} \times \vec{w}|^2 = \begin{vmatrix} \vec{v} \cdot \vec{v} & \vec{v} \cdot \vec{w} \\ \vec{w} \cdot \vec{v} & \vec{w} \cdot \vec{w} \end{vmatrix} = |\vec{v}|^2 |\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2$ (Exercise 1.3.5 Oprea, new HW)

$\Rightarrow$ Theorem (i), since

$$|\vec{v}|^2 |\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2 = |\vec{v}|^2 |\vec{w}|^2 (\underbrace{1 - \cos^2\theta}_{\sin^2\theta})$$

3
Return to Wed notes, begin by computing $\vec{\alpha}'(t)$, $\vec{\alpha}''(t)$, $\vec{\alpha}'(t) \times \vec{\alpha}''(t)$ for helix, and drawing these vectors at $t = \pi/2$.