Math 4530
Wed 12 Jan

<u>Curves</u> : A curve in 3-space (or 2-space) is a <u>continuous</u> function ("map") with domain some interval $I \subset \mathbb{R}$

$\alpha : I \longrightarrow \mathbb{R}^3$

<u>Def</u> $\vec{\alpha}$ is differentiable iff $\vec{\alpha}'(t)$ exists $\forall t \in I$ [$\alpha'(t)$ is called <u>tangent</u>, or <u>velocity</u> vector]

i.e.

$$\vec{\alpha}'(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\vec{\alpha}(t+\Delta t) - \vec{\alpha}(t)\right) \text{ exists}$$

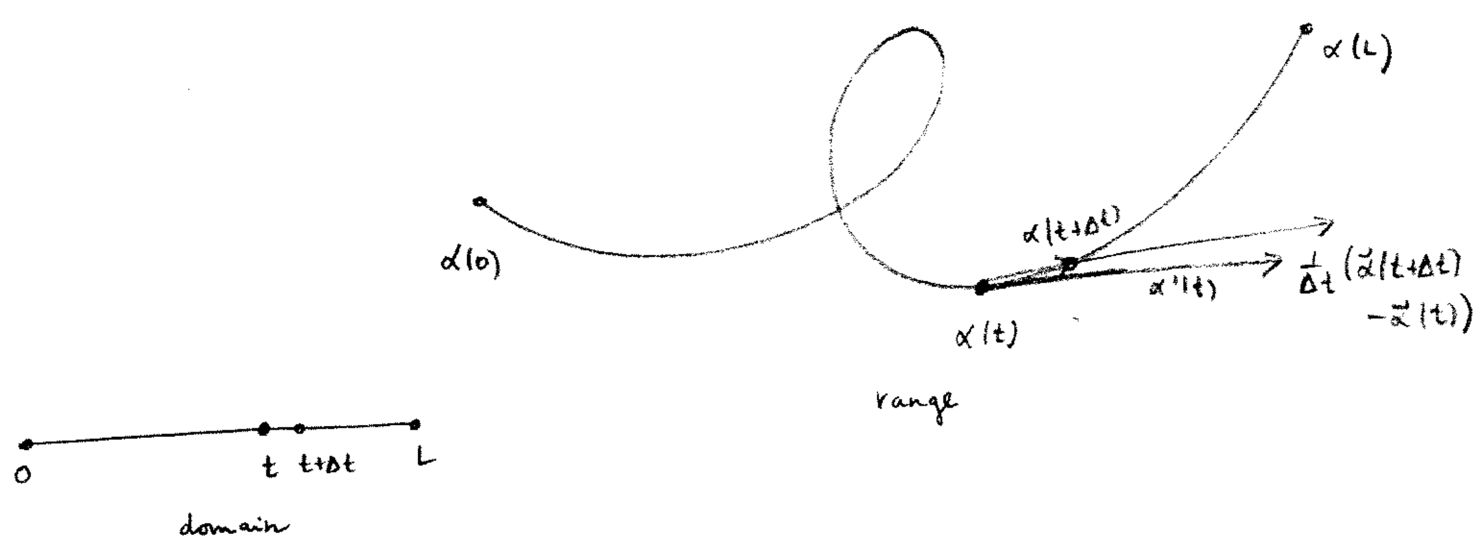
← note superscript notation – these are NOT powers!

Note, if $\vec{\alpha}(t) = \begin{bmatrix} \alpha^1(t) \\ \alpha^2(t) \\ \alpha^3(t) \end{bmatrix}$

$$\frac{1}{\Delta t}\left[\vec{\alpha}(t+\Delta t) - \vec{\alpha}(t)\right] = \frac{1}{\Delta t}\begin{bmatrix} \alpha^1(t+\Delta t) - \alpha^1(t) \\ \alpha^2(t+\Delta t) - \alpha^2(t) \\ \alpha^3(t+\Delta t) - \alpha^3(t) \end{bmatrix} = \begin{bmatrix} \frac{\alpha^1(t+\Delta t) - \alpha^1(t)}{\Delta t} \\ \frac{\alpha^2(t+\Delta t) - \alpha^2(t)}{\Delta t} \\ \frac{\alpha^3(t+\Delta t) - \alpha^3(t)}{\Delta t} \end{bmatrix}$$

Deduce (Math $\overset{2210}{3220}$), since vector limit exists iff all component limits exist

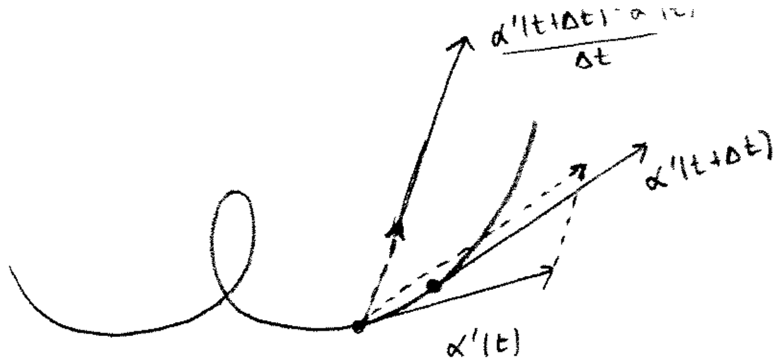
$$\vec{\alpha}'(t) = \begin{bmatrix} \frac{d\alpha^1}{dt} \\ \frac{d\alpha^2}{dt} \\ \frac{d\alpha^3}{dt} \end{bmatrix}$$



$\vec{\alpha}''(t) := \frac{d}{dt}\vec{\alpha}'(t)$

$\uparrow$

called acceleration vector

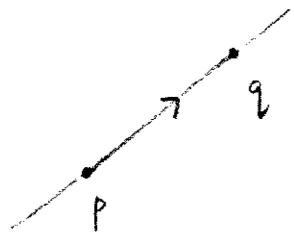


examples

$\alpha(t) = p + t(q-p)\ ,\ t \in \mathbb{R}$

$\alpha'(t) =$

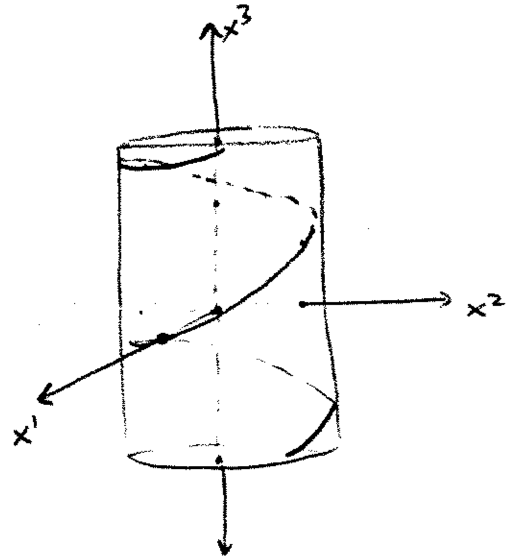
$\alpha''(t) =$



$\alpha(t) = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix}$; a helix

$\alpha'(t) =$

$\alpha''(t) =$



plot $\vec{\alpha}(\pi/2)$

$\vec{\alpha}'(\pi/2)$

$\vec{\alpha}''(\pi/2)$

Length : If $\alpha : \overset{= [a,b]}{I} \to \mathbb{R}$ is differentiable

Def 1 $\quad L(\alpha) := \int_a^b \underbrace{\|\alpha'(t)\|}_{\uparrow \text{ speed}} dt$

Length can be defined for any continuous curve

partition P

Def 2 $\quad L = \lim_{\|P\| \to 0} \underbrace{\sum |\alpha(t_{i+1}) - \alpha(t_i)|}_{\text{approx length}} = \sup_P \sum |\alpha(t_{i+1}) - \alpha(t_i)|$

This limit always exists. (may be $+\infty$)

Why? Use Δ inequality: $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$ (which you reprove in HW)

If α is continuously differentiable, it is relatively easy to show Def 1 & Def 2 agree.

One of your HW exercises is to show that if you reparameterize a curve then you get the same length (using Def 1).

Example

What is the length of one "loop" of the helix

$$\vec{\alpha}(t) = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix} \qquad 0 \le t \le 2\pi \text{ ?}$$

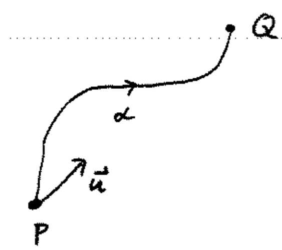
Theorem (You probably thought you knew this).

Let $P, Q \in \mathbb{R}^3$.

Then the shortest path from P to Q is a straight line.

proof 1 (from text). Let $\vec{\alpha}(t)$ connect P to Q,

$\alpha : [a,b] \to \mathbb{R}^3$.



$$L = \int_a^b |\alpha'(t)|\, dt$$

$$* \quad \geq \int_a^b \alpha'(t) \cdot \vec{u} \;\, dt \qquad \vec{u} = \frac{\vec{PQ}}{|\vec{PQ}|}$$

↑ dot prod.

$$= \left(\int_a^b \alpha'(t)\,dt\right) \cdot \vec{u}$$

$$= (\alpha(b) - \alpha(a)) \cdot \vec{u}$$

$$= (\vec{Q} - \vec{P}) \cdot \frac{(\vec{Q} - \vec{P})}{|\vec{Q} - \vec{P}|} = |\vec{Q} - \vec{P}| = \text{straight-line distance}$$

e.g. $\beta(t) = P + t(Q-P) \qquad 0 \le t \le 1$

$\beta'(t) = Q - P$

$\int_0^1 |\beta'(t)|\,dt = |\vec{Q} - \vec{P}|$ ✓.

* Note equality here iff $\alpha'(t) \parallel \vec{u} \;\; \forall t$ (by Cauchy Shwarz).

i.e. $\alpha'(t) = c(t)\vec{u}$

$\Rightarrow \alpha(t) = P + \int_0^t c(t)\,\vec{u} = P + b(t)\,\vec{u}$

$\Rightarrow \alpha(t)$ parameterizes straight line ∎

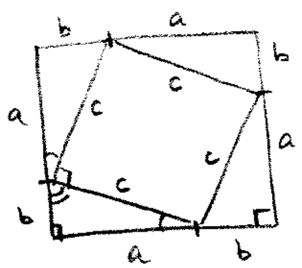
proof 2 Use Def 2 on page 3.

For the partion of $[a,b]$ consisting of $[a,b]$ itself, the approx. length is $|Q-P|$. When you refine the partion the approx lengths increase (unless $\alpha(t)$ moves monotonically along $\vec{PQ}$.) ∎

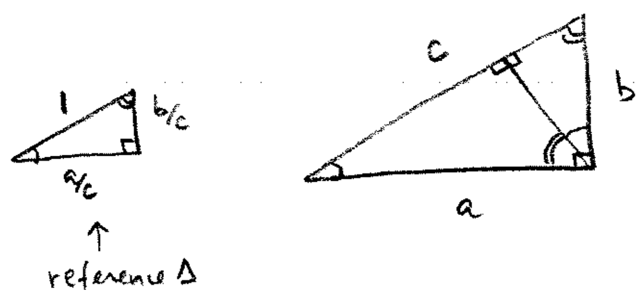
Homework due Friday 1/21
(More likely)

I Pythagorean Thm: $a^2+b^2=c^2$ for right triangles.

a) Prove P.T. from this diagram, by expressing the total area two ways:
(First use geometry to verify that inside c c is actually a square.)



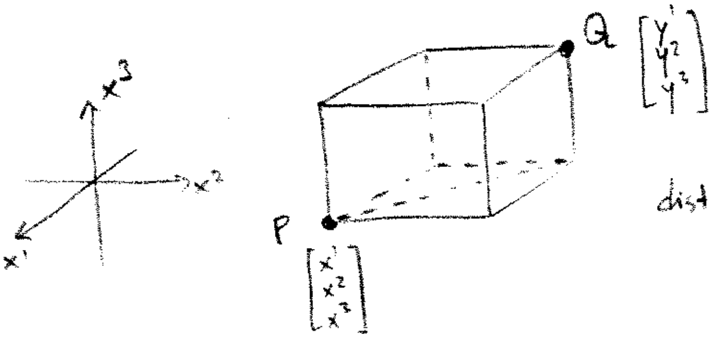
b) Give a proof of P.T. based on the fact that area scales by λ^2 when you dilate $\mathbb{R}^2$ by a factor of λ



← compute total area in terms of reference Δ area (two ways)

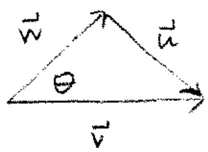
c) Use P.T. twice to prove $\mathbb{R}^3$ distance formula

(this is essentially exercise 1.4 p.3)



$$\text{dist}(P,Q) = \sqrt{(y^1-x^1)^2+(y^2-x^2)^2+(y^3-x^3)^2}$$

Id) Complete the proof of Proposition 1.8 page 6 : $\vec{w}\cdot\vec{v} = \|\vec{w}\|\|\vec{v}\|\cos\theta$



$\vec{u}$ $\vec{v}-\vec{w}$

$$\|\vec{u}\|^2 = \vec{u}\cdot\vec{u} = (\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w})$$
$$= \|\vec{v}\|^2 - 2\vec{v}\cdot\vec{w} + \|\vec{w}\|^2 \quad \text{(as in text)}$$

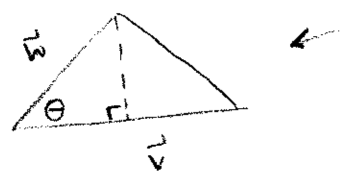
then text quotes law of cosines

$$* \quad \|\vec{u}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos\theta$$

from which prop. follows

Your job is to prove law of cosines from Pythag thm.

Hint : if $\theta < \pi/2$, $\|\vec{w}\| \leq \|\vec{v}\|$ use 2 P.T.'s ω diagram



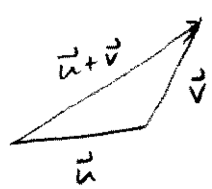
then treat cases $\theta > \pi/2$
$\theta = \pi/2$.

(equality iff $\vec{w} /\!/ \vec{v}$)

Ie) Since $\vec{w}\cdot\vec{v} = \|\vec{w}\|\|\vec{v}\|\cos\theta$, we know $\boxed{|\vec{w}\cdot\vec{v}| \leq \|\vec{w}\|\|\vec{v}\|}$ (take abs vals)

This is known as Cauchy-Schwarz inequality.

Use Cauchy-Schwarz to prove the triangle inequality



$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

equality iff $\vec{u}, \vec{v}$ are positive scalar mults of eachother

II Book exercises, chapter 1

1.1.13 (page 9), 1.1.14, 1.1.21, 1.1.22
1.2.2, 1.2.7, 1.2.8