Math 4530-1
Monday 1/10
LCB 219 10:45-11:35

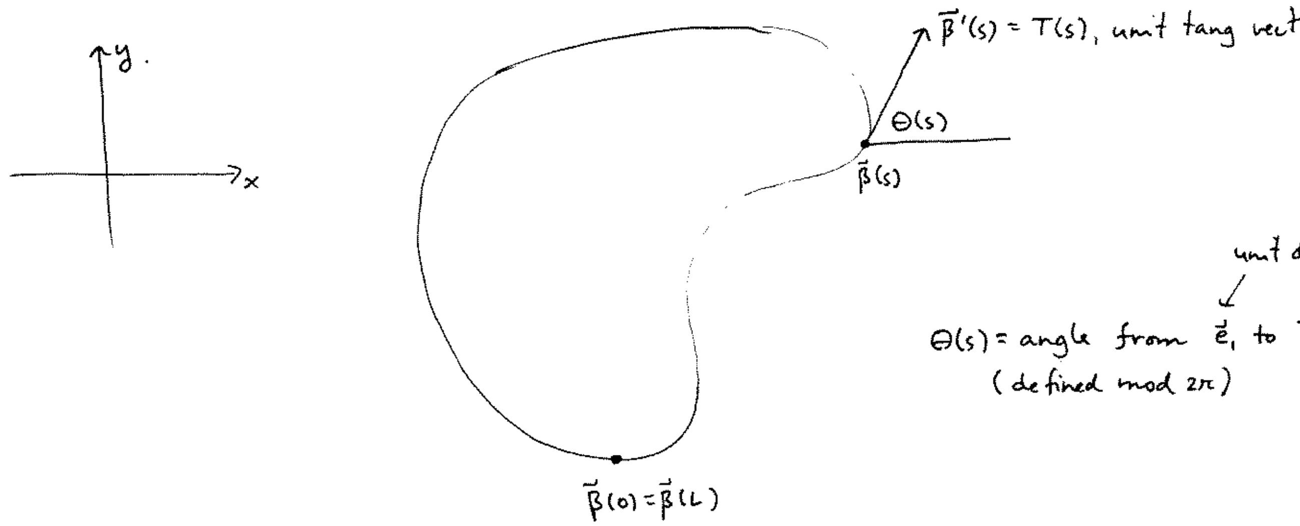
- homework: look over text intro & chapter 1
  I'll have some actual HW problems on Wed
- meeting times: no problem session this week. So pick one of
  Thurs 1/13 10, 10:20, 10:40, 11:00, 11:20
  Fri 1/14 1, 1:20, 1:40
  or we can arrange a time next week.

Two interesting global theorems:

"no corners" — no self-intersections, i.e. β is 1-1 except at endpoints — $\beta(0)=\beta(L)$ — image in plane

① Let $\vec{\beta}:[0,L]\to\mathbb{R}^2$ be a regular, simple closed plane curve parameterized by arclength

$|\beta'(s)|=1$, "unit speed"



unit dir of pos x-axis

$\Theta(s)$ = angle from $\vec{e}_1$ to $T(s)$
(defined mod 2π)

planar curvature $\kappa(s) := \Theta'(s)$

$$\boxed{\text{Theorem: } \int_0^L \kappa(s) = \pm 2\pi}$$

not too surprising.

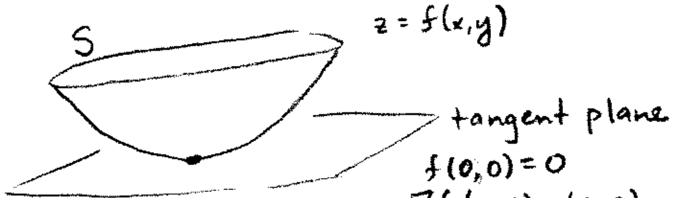
example: Circle of radius R, counterclockwise

$\alpha(t) = R\begin{bmatrix}\cos t\\ \sin t\end{bmatrix}$

$\beta(s) = R\begin{bmatrix}\cos(s/R)\\ \sin(s/R)\end{bmatrix}$ $|\beta'|\equiv 1$ (?)

$\Theta(s) = s/R$
$\Theta'(s) = 1/R$ $\int_0^{2\pi R} \frac{1}{R}\,ds = 2\pi$ ✓

More amazing Thm for surfaces: rotate & translate S to move fixed pt to origin, and make tangent plane horizontal.



$f(0,0)=0$

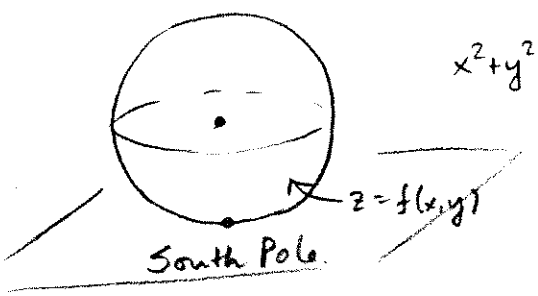
$\nabla f(0,0)=(0,0)$

$$D^2 f(0,0) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \qquad \text{(Matrix of shape operator)}$$

invariants: $\frac{1}{2}\,\text{trace} = \frac{1}{2}(f_{xx}+f_{yy})$ "Mean curvature" H $\quad \frac{1}{2}(\lambda_1+\lambda_2)$

$\det = f_{xx}f_{yy} - f_{xy}^2$ "Gauss curvature" K $\quad \lambda_1\lambda_2$

Example: sphere of radius R



$f(\vec{0})=0$

$\nabla f(\vec{0})=0$

$$x^2+y^2+(z-R)^2 = R^2$$

$2x + 2(f-R)f_x = 0$ $\qquad x=0,\ f=0 \Rightarrow f_x=0$

$2y + 2(f-R)f_y = 0$ $\qquad y=0 \Rightarrow f_y=0$

$2 + 2f_x^2 + 2(f-R)f_{xx} = 0$ $\qquad x=y=f=0 \Rightarrow f_{xx}=\frac{1}{R}$

$2f_y f_x + 2(f-R)f_{xy} = 0$ $\qquad f_{xy}=0$

$2 + 2f_y^2 + 2(f-R)f_{yy} = 0$ $\qquad f_{yy}=\frac{1}{R}$

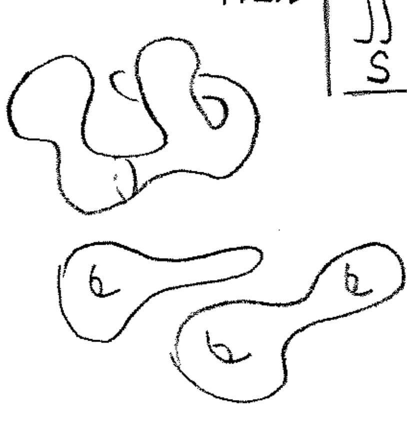
$$D^2 f(0,0) = \begin{bmatrix} 1/R & 0 \\ 0 & 1/R \end{bmatrix} \qquad H = 1/R \quad K = 1/R^2$$

<u>Theorem</u> (Gauss-Bonnet) Let $S \subset \mathbb{R}^3$ be an embedded regular compact surface (without boundary)

Then $\boxed{\iint_S K\,dA = 2\pi\chi(S)}$ surprising!

$\chi(S)$: Euler characteristic of S

$V-E+F$ for a triangulation of S $\quad (= 2-2g,\ g = \text{genus})$



for sphere

$\chi(S) = 4-6+4 = 2$

$\iint_S K\,dA = 4\pi$ for spheres (of any shape)

$= 0$ for tori (of any shape)

etc.