Math 4530
Wed 2/9

- page 11 – directional derivatives on surfaces

- Remark on HW related to page 11:

  If $f: M^2 \to \mathbb{R}$ then

  $$\vec{X}_u[f] = (f \circ \vec{X})_u \qquad \text{so} \quad \vec{X}_v\left[\vec{X}_u[f]\right] = \vec{X}_v\left[(f\circ\vec{X})_u\right] = (f\circ\vec{X})_{uv}$$
  $$\vec{X}_v[f] = (f\circ\vec{X})_v \qquad\qquad = (f\circ\vec{X})_{vu} = \vec{X}_u\left[\vec{X}_v[f]\right]$$

  In Exercise 2.2.7 you verify this equality of cross partials assuming that f is the restriction of a function $f: \mathbb{R}^3 \to \mathbb{R}$ to M^2 and using the chain rule.

  Exercise 2.2.9 is the analogous computation for covariant derivatives of vector fields; since

  $$\nabla_{\vec{X}_u} \vec{F} = (\vec{F}\circ\vec{X})_u$$

  we must have $\nabla_{\vec{X}_v}(\nabla_{\vec{X}_u}\vec{F}) = \nabla_{\vec{X}_u}(\nabla_{\vec{X}_v}\vec{F})$.

- The Shape operator

Let $\vec{U}$ a unit normal choice on M^2

For $p \in M$

$\vec{v} \in T_pM$

$S(\vec{v}) := -\nabla_{\vec{v}}\vec{U}$ (the opposite of the directional derivative of $\vec{U}$)

$S(\vec{v}) \in \mathbb{R}^3$

In fact, $S(\vec{v}) \in T_pM$, since $\vec{U}\cdot\vec{U} \equiv 1$

$$\Rightarrow \vec{v}(\vec{U}\cdot\vec{U}) = 0$$
$$\Rightarrow 2(\nabla_{\vec{v}}\vec{U}\cdot\vec{U}) = 0$$
i.e. $\nabla_{\vec{v}}\vec{U} \perp \vec{U}$.

So $S: T_pM \to T_pM$ is a linear map. It encodes the way M is bending at p.

We will use linear algebra to understand the shape operator, along with what we learned about curvature of curves

If $X : D \to M$ is a local patch,

then $S(\vec{X}_u) = -\nabla_{X_u}(\vec{U}) = -(\vec{U}\circ\vec{X})_u$ (we often just write U_u)

$S(\vec{X}_v) = -(\vec{U}\circ\vec{X})_v$

Examples

① plane: $\vec{X}(u,v) = u\vec{f}_1 + v\vec{f}_2 + \vec{P}$. Choose $\{\vec{f}_1, \vec{f}_2\}$ orthonormal

Choose $U = \vec{f}_1 \times \vec{f}_2$ constant

$\Rightarrow \vec{U}_u = \vec{U}_v \equiv 0$

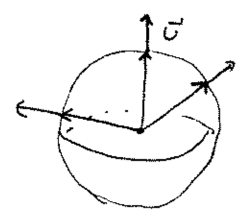
$\Rightarrow S_p \equiv 0$ (the zero linear map)

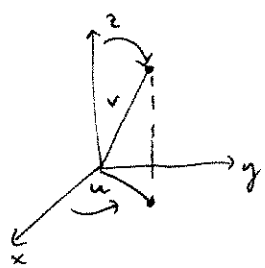
② sphere of radius R

spherical coords

$$X(u,v) = R\begin{bmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{bmatrix}.$$

Choose $\vec{U}(u,v) = \frac{1}{R}\vec{X}(u,v)$





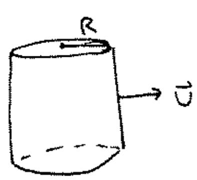
$S(\vec{X}_u) = -\vec{U}_u = -\frac{1}{R}\vec{X}_u$

$S(\vec{X}_v) = -\vec{U}_v = -\frac{1}{R}\vec{X}_v$

so S_p is dilation ~~with respect to~~ by $-\frac{1}{R}$

$$[S]_{\{\vec{X}_u, \vec{X}_v\}} = \begin{bmatrix} -\frac{1}{R} & 0 \\ 0 & -\frac{1}{R} \end{bmatrix}$$

③ cylinder of radius R



$$\vec{X}(u,v) = \begin{bmatrix} R\cos u \\ R\sin u \\ v \end{bmatrix} \qquad \vec{X}_u = R\begin{bmatrix} -\sin u \\ \cos u \\ 0 \end{bmatrix}, \quad \vec{X}_v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{U}(u,v) = \begin{bmatrix} \cos u \\ \sin u \\ 0 \end{bmatrix}$$

$$S(\vec{X}_u) = -\vec{U}_u = \begin{bmatrix} -\sin u \\ \cos u \\ 0 \end{bmatrix} = -\frac{1}{R}\vec{X}_u \qquad [S]_{\{\vec{X}_u, \vec{X}_v\}} = \begin{bmatrix} -1/R & 0 \\ 0 & 0 \end{bmatrix}$$

$S(\vec{X}_v) = -\vec{U}_v = 0$

Recall Linalg:

$S: T_pM \to T_pM$ linear

pick a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ for T_pM

If $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$, $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ coordinates of $\vec{v}$ w.r.t. $\mathcal{B}$.

Then

$S(\vec{v}) = c_1 S(\vec{v}_1) + c_2 S(\vec{v}_2)$ linearity.

so $[S(\vec{v})]_{\mathcal{B}} = c_1 [S(v_1)]_{\mathcal{B}} + c_2 [S(v_2)]_{\mathcal{B}}$ since taking coords wrt $\mathcal{B}$ is a linear map from T_pM to $\mathbb{R}^2$.

$$= \underbrace{\left[\begin{array}{c:c} [S(\vec{v}_1)]_{\mathcal{B}} & [S(v_2)]_{\mathcal{B}} \end{array}\right]}_{} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

this is called the matrix of S w.r.t. $\mathcal{B}$, written $[S]_{\mathcal{B}}$.
First column is coords of $S(\vec{v}_1)$
2^{nd} col is coords of $S(\vec{v}_2)$
These are the matrices we wrote on page 13.

What properties of S
do <u>not</u> depend on which basis we
choose for T_pM? (i.e. on our local choice of chart).

<u>Ans</u> The eigenvalues and eigenvectors; i.e. the eigenpairs $(\lambda, \vec{v})$ so that

$$S\vec{v} = \lambda\vec{v}$$

the eigenvalues of S are called <u>principal curvatures</u>
the eigenvectors are called <u>principal directions</u>
They each have geometric meaning.

What are they in the page 13 examples?