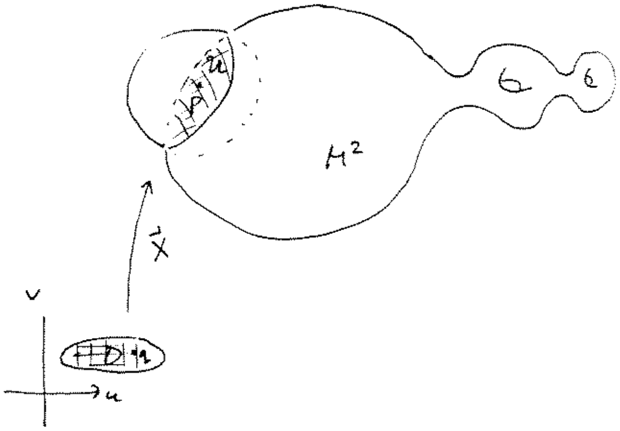
Math 4530
Monday 2/7

Recall surface ideas from Friday:

- $M^2 \subset \mathbb{R}^3$ is a differentiable surface
  iff $\forall\, p \in M\ \exists\ \mathcal{U}_{\text{open}},\ p \subset \mathcal{U} \subset M$
  s.t. $\mathcal{U} = \vec{X}(D)$, $\vec{X}: D \to \mathbb{R}^3$ a patch
  $\begin{pmatrix} \vec{X} \text{ diff'ble} \\ \vec{X}_u \times \vec{X}_v \neq \vec{0} \\ \vec{X}\ 1\text{-}1 \end{pmatrix}$



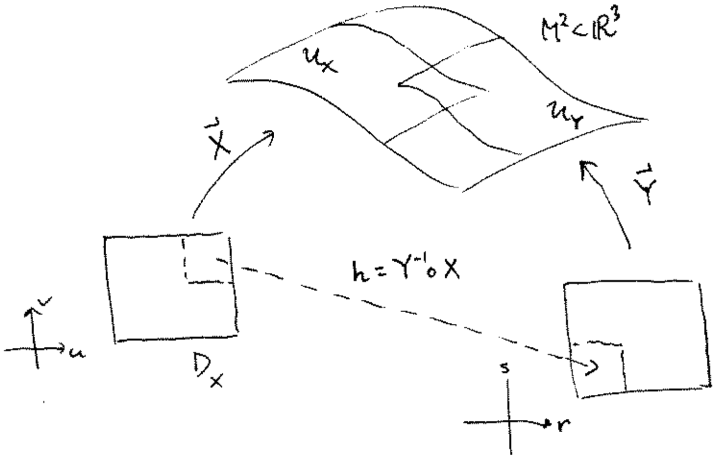
- A collection of $\overset{\text{charts}}{\text{patches}}$ is an atlas

- You show in HW (see II page 7) that if you have two charts $\vec{X}: D_X \to \mathcal{U}_X \subset M \subset \mathbb{R}^3$
  $\vec{Y}: D_Y \to \mathcal{U}_Y \subset M \subset \mathbb{R}^3$

  then the transition functions

  $$h = Y^{-1}\circ X : D_X \cap X^{-1}(\mathcal{U}_Y) \to D_Y \cap Y^{-1}(\mathcal{U}_X)$$

  and its inverse, $X^{-1}\circ Y$: (reverse order)
  are differentiable



- Recall also, how all differentiability questions are defined via patches, for curves, for functions to $\mathbb{R}$, and for maps between surfaces (page 5).

This leads to a compatibily of defintions question for curves $\alpha: I \to M \subset \mathbb{R}^3$, which you answer in another HW exercise (I, page 7).

- Discuss the tangent space of M at p, denoted by $T_p(M)$ — page 6

- <u>Unit normal & orientability</u>

If M is a surface and $X: D \to \mathbb{R}^3$ is a patch, we get two possible unit normals

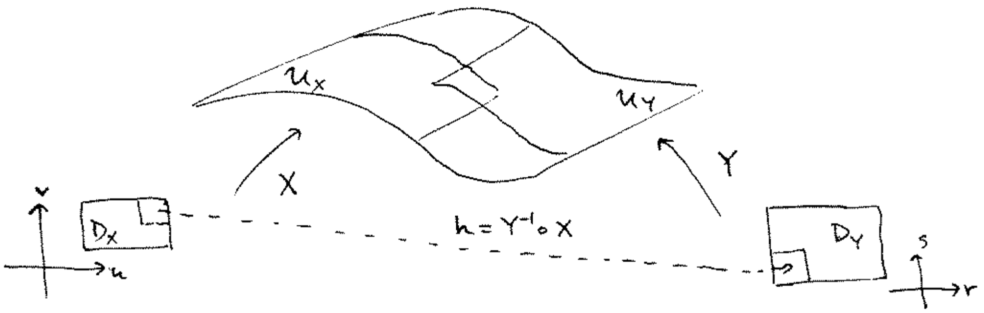
$$U := \pm \frac{X_u \times X_v}{|X_u \times X_v|}$$

<u>Def</u>: M^2 is orientable iff it is possible to choose a unit normal <u>continuously</u> on M,

i.e. $\exists\, \mathcal{U}: M \to \underset{\substack{\uparrow \\ \text{unit} \\ \text{sphere}}}{S^2}$ continuous, s.t. $U(p) \perp T_pM \quad \forall p \in M$.

<u>Example</u>: S^2 is orientable, the Möbius strip is not ("proof": look at the Möbius strip).

How would a cartographer describe orientability?
You must look at how charts glue together for the answer:



$$Y^{-1} \circ X = h$$

so $$X = Y \circ h$$

<u>chain rule</u> (better get comfortable with this!) :
$$\begin{bmatrix} X_u & \vdots & X_v \end{bmatrix} = \begin{bmatrix} Y_r & \vdots & Y_s \end{bmatrix} \begin{bmatrix} r_u & r_v \\ s_u & s_v \end{bmatrix}$$

so $$X_u \times X_v = [Y_r r_u + Y_s s_u] \times [Y_r r_v + Y_s s_v] = (Y_r \times Y_s) \underbrace{(r_u s_v - s_u r_v)}_{\det[Dh]}$$

<u>Deduce</u> ① If M is orientable you can (re-)draw atlas s.t. $U = \frac{X_u \times X_v}{|X_u \times X_v|}$ for <u>every</u> chart, by interchanging variables if necessary

this def generalizes to diff'ble manifolds M^n.

② If a cartographer can create an atlas s.t. all $\det[Dh] > 0$, then we can define continuous $U := \frac{\vec{X}_u \times \vec{X}_v}{|X_u \times X_v|}$ globally on M

Directional derivatives in Euclidean Space

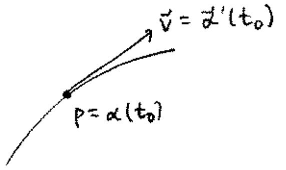
in multivariable Calc you study directional derivatives.
Do you remember the formula

$$D_{\vec v} f(p) = \nabla f(p)\cdot \vec v = \langle f_{x_1}(p), f_{x_2}(p), \dots f_{x_n}(p)\rangle \cdot \langle v_1, v_2, \dots v_n\rangle \qquad ?$$

Geometer's notation:
We denote this directional derivative by $\vec v[f]$

or $\underset{\substack{\uparrow\\ \text{vector}}}{\vec v}[\underset{\substack{\uparrow\\ \text{fcn}}}{f}](\underset{\substack{\nwarrow\\ \text{point.}}}{p})$

Note, if $\vec\alpha$ is any curve with $\alpha(t_0)=p$
$\alpha'(t_0)=v$



then

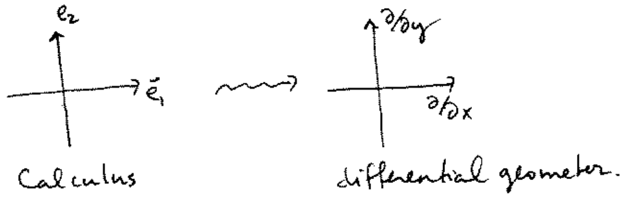
$$\frac{d}{dt} f(\alpha(t))\Big|_{t=t_0} = [f_{x_1}(p), \dots f_{x_n}(p)]\begin{bmatrix}\alpha^{1\prime}(t_0)\\ \alpha^{2\prime}(t_0)\\ \vdots \\ \alpha^{n\prime}(t_0)\end{bmatrix} = \vec v[f](p)$$

Thus

Def: $\vec v[f](p) := \frac{d}{dt} f(\alpha(t))\Big|_{t=t_0}$ α any curve s.t. $\alpha(t_0)=p$
$\alpha'(t_0)=v$

Example: $f(x,y)$ $\vec v = \begin{bmatrix}1\\0\end{bmatrix}$ $\vec v[f]\begin{pmatrix}x\\y\end{pmatrix} = \frac{d}{dt} f\begin{pmatrix}x+t\\y\end{pmatrix}\Big|_{t=0} = f_x$

$\vec w = \begin{bmatrix}0\\1\end{bmatrix}$ $\vec w(f)\begin{pmatrix}x\\y\end{pmatrix} = f_y$



properties

$(\vec v + \vec w)(f) = \vec v(f) + \vec w(f)$
$(c\vec v)(f) = c\,\vec v(f)$ } directional derivs are linear wrt direction

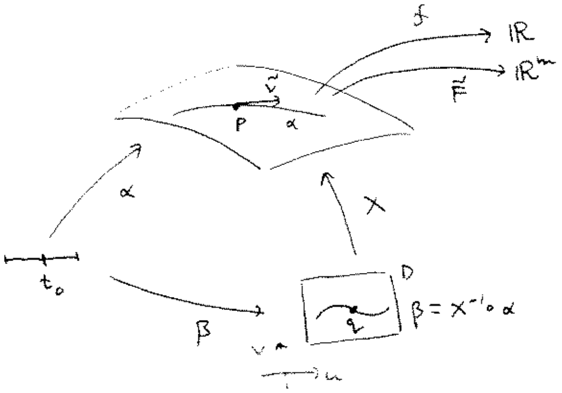
$\vec v(fg) = (\vec v(f))g + f\,\vec v(g)$ } product rule. You can guess, after curve theory, that this gets a lot of use.

If $\vec F, \vec G : \mathbb{R}^n \to \mathbb{R}^m$ then define

$$\nabla_{\vec v}\vec F = \begin{bmatrix} v(F^1)\\ v(F^2)\\ \vdots\\ v(F^m)\end{bmatrix} = \frac{d}{dt}\vec F(\alpha(t))\Big|_{t=t_0}$$

. Note $\vec v(\vec F\cdot\vec G) = (\nabla_{\vec v}\vec F)\cdot\vec G + \vec F\cdot(\nabla_{\vec v}\vec G)$.

Directional Derivatives on surfaces



Use curve def'n

$\alpha : I \to M$
$\alpha(t_0) = p$
$\alpha'(t_0) = \vec{v}$

$$\vec{v}[f](p) := \frac{d}{dt} f\circ\alpha(t)\Big|_{t=t_0}$$

To see that this definition is well-defined,
and to see how it works out computationally,
check things out in a local patch $X : D \to \underset{\overset{\psi}{p}}{\mathcal{U}} < M$:

$$X^{-1}\circ\alpha = \beta$$
$$\alpha = X\circ\beta$$
$$\alpha' = X_u u' + X_v v'$$

So, If $\alpha'(t_0) = aX_u + bX_v$ $\quad (X_u(q), X_v(q))$
Then $\beta'(t_0) = \langle a, b\rangle$.

So, $$\vec{v}[f](p) = \frac{d}{dt} f\circ\alpha(t)\Big|_{t=t_0} = [f\circ(X\circ\beta)]'(t_0) = [(f\circ X)\circ\beta]'(t_0)$$
$$= (f\circ X)_u a + (f\circ X)_v b$$

Thus
- $\vec{v}[f](p)$ is well-defined.
- Linearity and product rule properties page 10 hold.
- Special cases
$$\boxed{\begin{array}{l} X_u(f) = (f\circ X)_u \\ X_v(f) = (f\circ X)_v \end{array}}$$
- If for $\vec{F} : M \to \mathbb{R}^n$, $\nabla_{\vec{v}}\vec{F}(p) := \begin{bmatrix} \vec{v}[f^1] \\ \vec{v}[f^2] \\ \vdots \\ \vec{v}[f^n] \end{bmatrix}$
then $\nabla_{\vec{v}}[\vec{F}\cdot\vec{G}] = (\nabla_{\vec{v}}\vec{F})\cdot\vec{G} + \vec{F}\cdot(\nabla_{\vec{v}}\vec{G})$

We will study the bending of
M in terms of the shape operator

U = unit normal choice on M

$S(\vec{v}) := -\nabla_{\vec{v}} U$

$S : T_pM \to \mathbb{R}^3$, linear map. In fact, $\boxed{S : T_pM \to T_pM \text{ linear.}}$