Math 4530
Friday 2/4
Surfaces!

Homework for 2/11

I, II From page 7 of today's notes

Also, from Chapter 2:

1.12, 1.13, ~~1.14~~ 1.15, 1.20, 1.21, 1.22

2.5, 2.6, 2.7, 2.8, 2.9

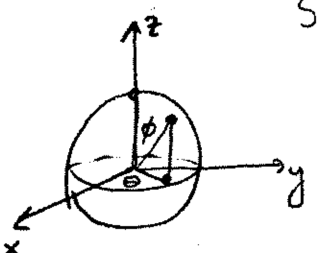
5.6, 5.7, 5.8 ← Maple

To describe where you are on a curve you need only 1 parameter [once you know you're on the curve]

$\vec{\alpha}(t)$
$\uparrow$

To describe where you are on a surface you need two coordinates, [once you know you're on the surface].

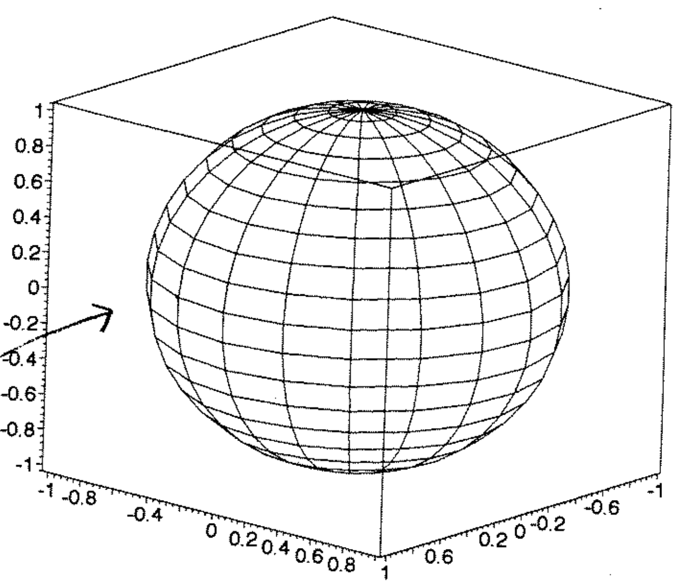
How we describe the surface of the earth leads to a lot of the vocabulary we use in studying surfaces. Consider spherical coords on $S^2 := \{\vec{x} \in \mathbb{R}^3 \mid \|\vec{x}\| = 1\}$



$u = \theta$
$v = \phi$

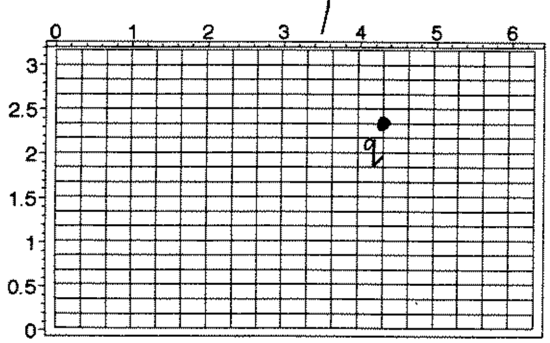
$$X\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{bmatrix}$$

image of spherical coord patch



u = latitude
v = longitude
[in radians]

domain of spherical coord patch



V (φ)

u ($=\theta$)

The father of all Maps!
mother?

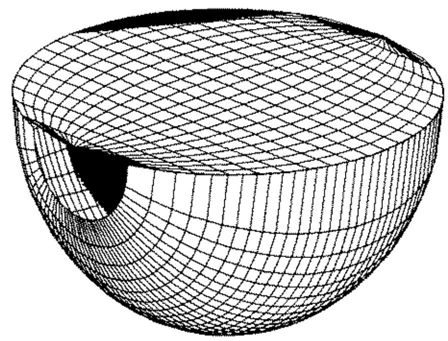
$$D = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} : 0 < u < 2\pi,\ 0 < v < \pi \right\}$$

$q = 120°$ W
$40°$ N

Monge patch for southern hemisphere

$$D = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} : u^2 + v^2 < 1 \right\}$$

$$X \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ v \\ -\sqrt{1-u^2-v^2} \end{bmatrix}$$



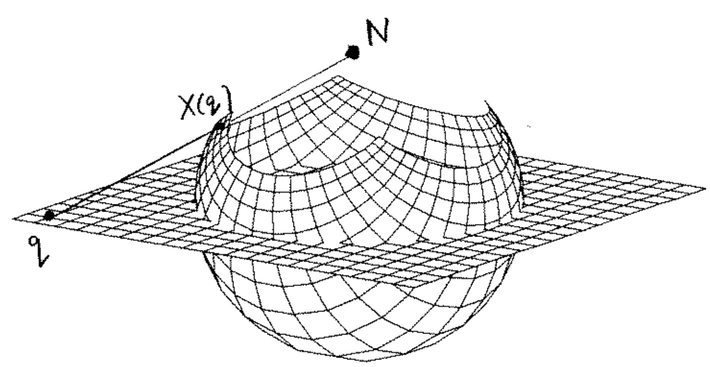
Which set of curves on sphere is <u>not</u> a set of u,v coord curves?

Every time you look at a Euclidean projection of the sphere onto a screen or piece of paper, this is a representation of a graphical patch, with respect to rectangular coords in the paper. Graphical patches are called "Monge patches" (in our book)

inverse stereographic projection patch

$$D = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} : \begin{matrix} -1.5 < u < 1.5 \\ -1.5 < v < 1.5 \end{matrix} \right\}$$

$$X \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{2u}{u^2+v^2+1} \\ \frac{2v}{u^2+v^2+1} \\ \frac{u^2+v^2-1}{u^2+v^2+1} \end{bmatrix}$$

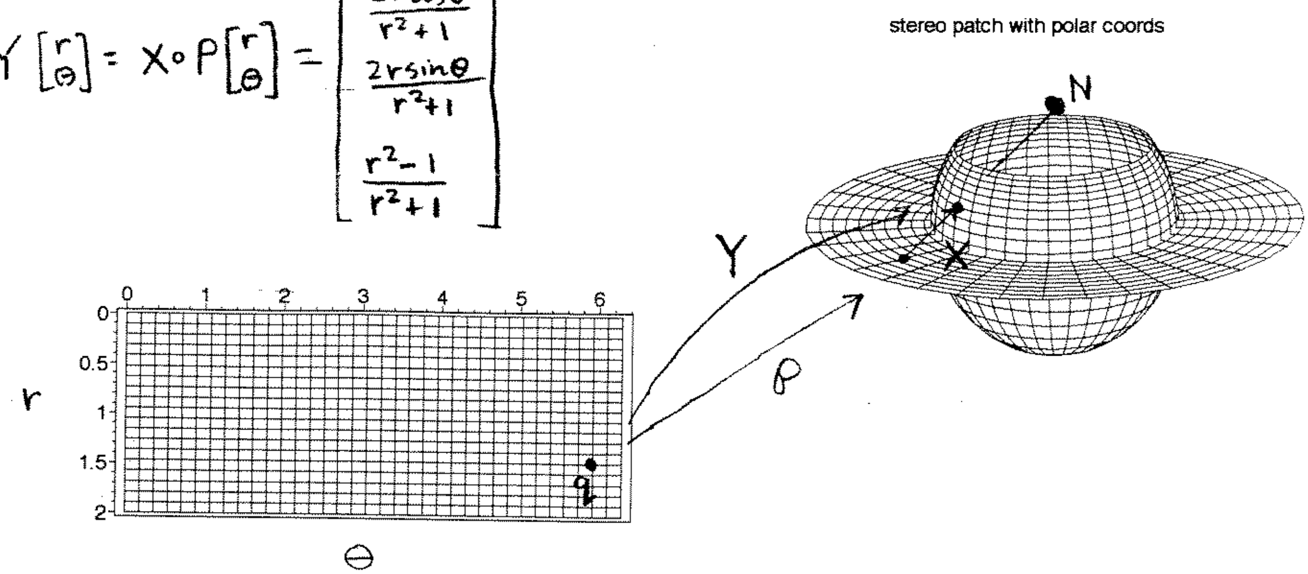


(interesting coordinate curves)

$$D = \left\{ \begin{bmatrix} r \\ \theta \end{bmatrix} : 0 < r < 2,\ 0 < \theta < 2\pi \right\}$$

$$\rho \begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

$$Y \begin{bmatrix} r \\ \theta \end{bmatrix} = X \circ \rho \begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} \frac{2r\cos\theta}{r^2+1} \\ \frac{2r\sin\theta}{r^2+1} \\ \frac{r^2-1}{r^2+1} \end{bmatrix}$$



General Language (for all surfaces)

$D \subset \mathbb{R}^2$ open, connected

$X : D \to \mathbb{R}^3$ differentiable (assume C^∞)

$$X \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x^1(u,v) \\ x^2(u,v) \\ x^3(u,v) \end{bmatrix}$$

X is <u>regular</u> iff $X_u \times X_v = \vec{0}$ $\forall\, u,v \in D$

X is $\left\{ \begin{array}{l} \text{(coordinate) patch} \\ \text{local parameterization} \\ \text{chart} \end{array} \right\}$ iff X is <u>regular</u> and <u>1-1</u>

$M^2 \subset \mathbb{R}^3$ is called a (differentiable) <u>surface</u> iff $\forall\, p \in M$ $\exists$ (open) neighborhood $\mathcal{U}$, $p \in \mathcal{U} \subset M$ s.t. $\mathcal{U}$ is the image of a patch $X : D \to \mathbb{R}^3$

A collection of patches (charts) which cover a surface M is called an <u>atlas</u> (really!)

- One of the principles of differential geometry (and cartography) is that you only need an atlas to study a surface, not an actual physical realization of the surface in space.

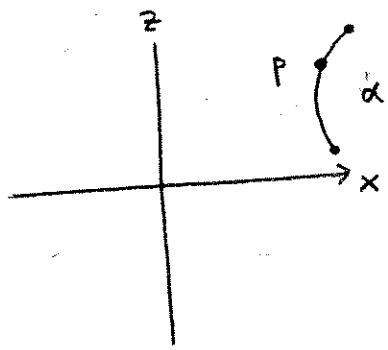
<u>coordinate curves</u> on the image of a patch $X\begin{bmatrix} u \\ v \end{bmatrix}$ are curves on which u or v are constant.

A Monge patch is a graphical patch ("above" some plane, often a coord plane in $\mathbb{R}^3$)

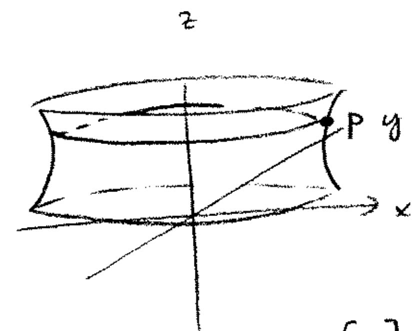
Surface Zoo

- graphs
- quadric surfaces
  (level sets of quadratic fcns in x, y, z)
- surfaces of revolution
  (and related objects like Möbius strip)
- ruled surfaces
  including cones, cylinders, helicoid, Möbius strip.

<u>surface of revolution</u> (e.g. about z-axis)



$$\alpha(t) = \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}$$



$p = \begin{bmatrix} a \\ 0 \\ b \end{bmatrix}$ is in circle $\begin{bmatrix} a\cos\theta \\ a\sin\theta \\ b \end{bmatrix}$ centered at $\begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix}$

$$X(t,\theta) = \begin{bmatrix} x(t)\cos\theta \\ x(t)\sin\theta \\ z(t) \end{bmatrix}$$

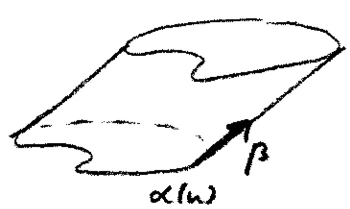
e.g. torus
almost e.g. Möbius

<u>ruled surface</u> : $X(u,v) = \alpha(u) + v\beta(u)$

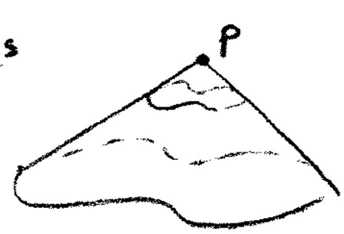
curve as u varies

line as v varies

e.g. • some quadrics
• <u>cylinders</u> : β constant :



• <u>cones</u>



$\beta(u) \parallel P - \alpha(u)$

(Start of) Differential Geometry of surfaces

overview

to study the geometry of curves we were able to

(1) pick a "best" parameterization (arclength s)

(2) study how the Frenet frame changed along the curve, by studying its s-derivative.

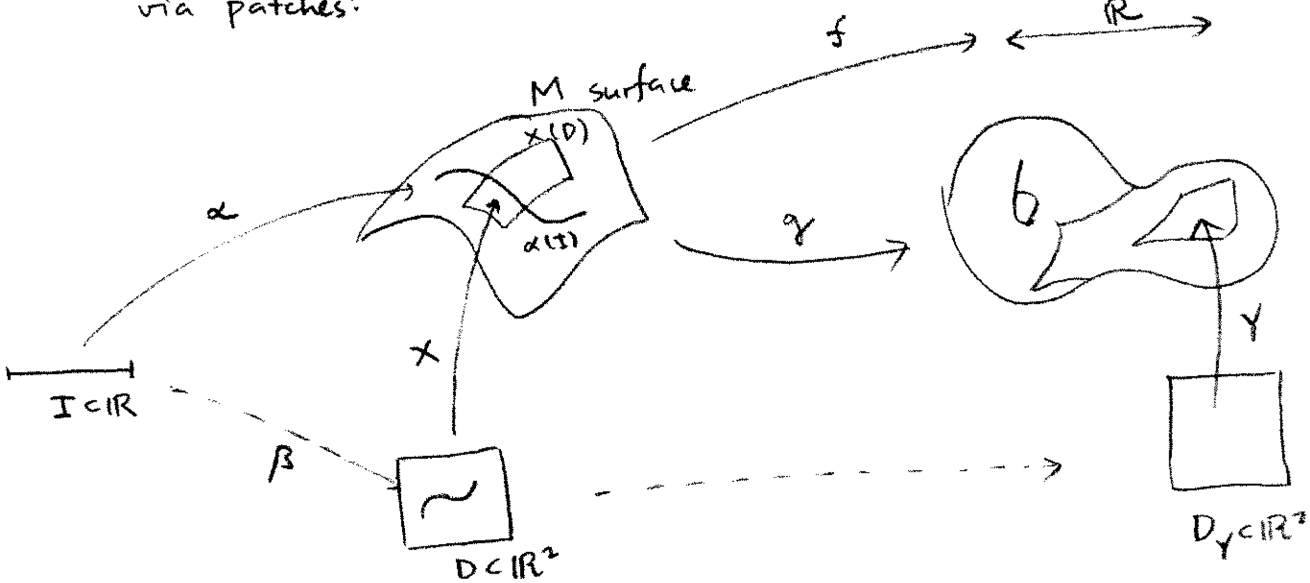
a surface will have only one "normal" direction (an advantage over curves), and if we take a unit normal vector U (locally O.K., on patch take $U = \frac{X_u \times X_v}{\| X_u \times X_v \|}$)

then the bending of the surface is described by how U changes along the surface. But there are disadvantages vis a vis curves:

(1) usually there is no "best" patch; in fact on many regular surfaces you need several patches which glue together awkwardly.

(2) You will need to study how U changes using directional derivatives (and curves on the surface), and what do we mean by a directional derivative on a surface anyway?

Cartographer/Geometer principle: differentiability questions are to be dealt with via patches:



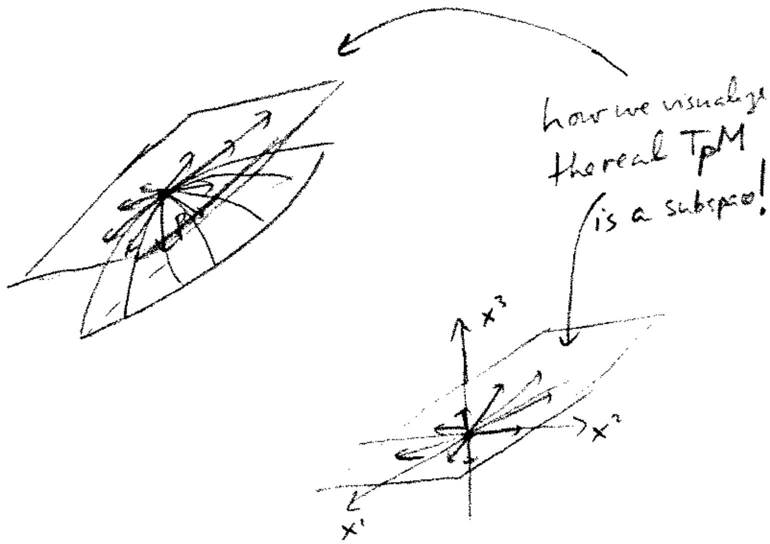
$\alpha : I \to M$ is diffble iff $X^{-1} \circ \alpha : \beta : I \to D$ diffble $\forall$ patches $X : D \to M$

$f : M \to \mathbb{R}$ (or $\mathbb{R}^n$) diffble iff $f \circ X : D \to \mathbb{R}$ (or $\mathbb{R}^n$) diffble $\forall$ patches $X : D \to M$

$g : M \to N$ diffble iff $Y^{-1} \circ g \circ X : D \to D_Y$ diffble $\forall$ patches $X : D \to X$, $Y : D_Y \to N$

T_pM

(tangent vectors and tangent spaces)



$\alpha : I \to M$ diffble curve

$\alpha(t_0) = p$

$\alpha'(t_0) = \vec{v}$

$T_pM = \{ \vec{v} \mid \vec{v} = \alpha'(t_0) \text{ as above} \}$

"tangent space at p to M"

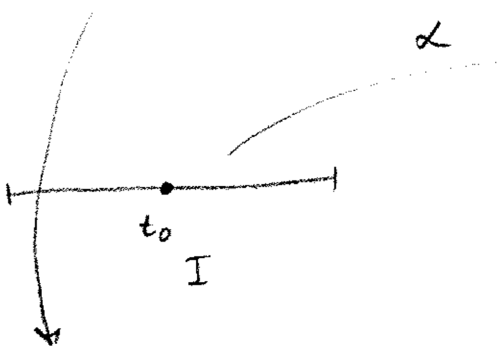
<u>Theorem</u>: Let $X : D \to M$ a patch

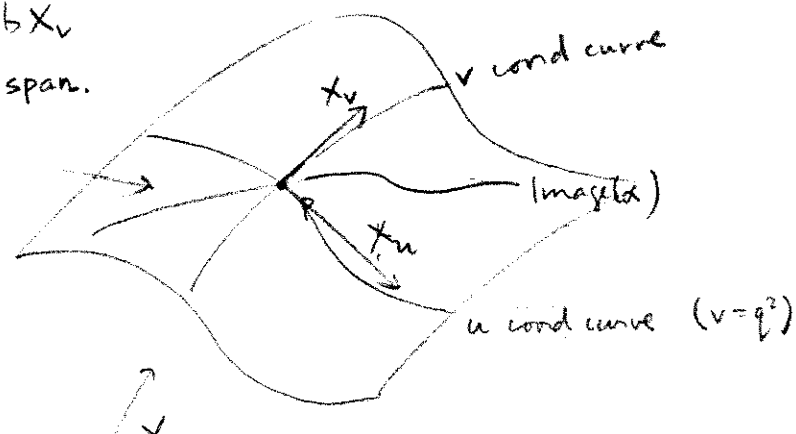
$X(q) = p$

then T_pM is a 2-dim'l vector space, with basis $\{X_u(q), X_v(q)\}$

pf: $\alpha(t) = X \circ \beta(t)$

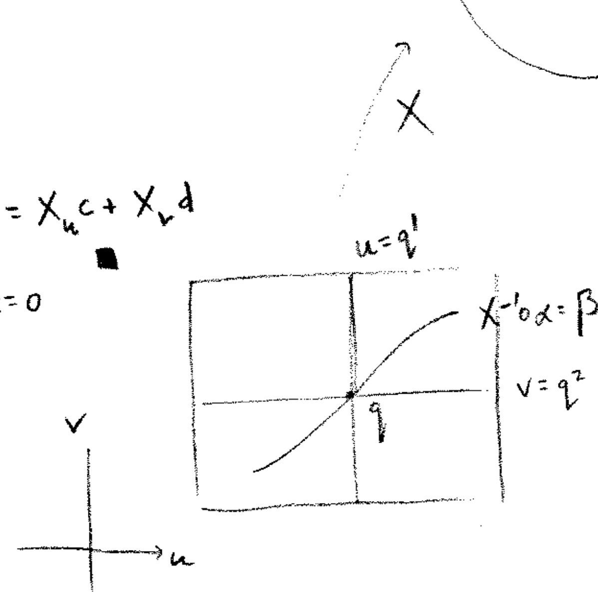
$\Rightarrow \alpha'(t_0) = \begin{bmatrix} X_u \mid X_v \end{bmatrix}_q \begin{bmatrix} \beta'(t_0) \end{bmatrix} = a X_u + b X_v$ shows span.





Converse:

$$\frac{d}{dt}\left(X\left(q + t\begin{bmatrix} c \\ d \end{bmatrix}\right)\right)\Bigg|_{t=0} = X_u c + X_v d \quad \blacksquare$$



<u>Cor</u> $T_pM = U^\perp$

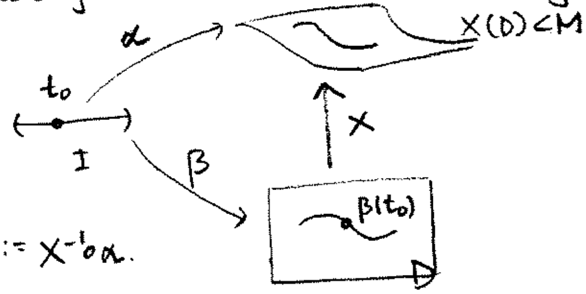
where $U = \dfrac{X_u \times X_v}{\|X_u \times X_v\|}$

is a unit normal.

Homework for 2/11

(I) If M is a surface and $\alpha : I \to M$ is a curve then we say α is differentiable iff $X^{-1} \circ \alpha : I \to D \subset \mathbb{R}^2$ is diff'ble $\forall$ patches (X, D) (and appropriate t). In this problem you are to show that this notion of differentiability is equivalent to saying that α is differentiable in the Chapter 1 sense, considered as a map to $\mathbb{R}^3$.

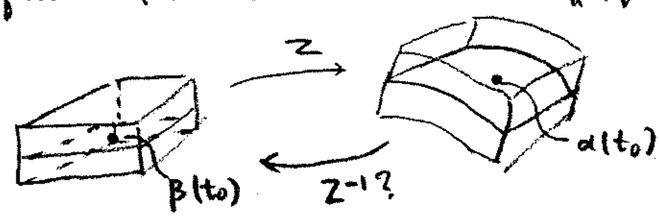
Hint: We may assume $\text{image}(\alpha) \subset X(D) \subset M$, by looking on a subinterval of the original one. Consider the diagram



$\beta := X^{-1} \circ \alpha$.

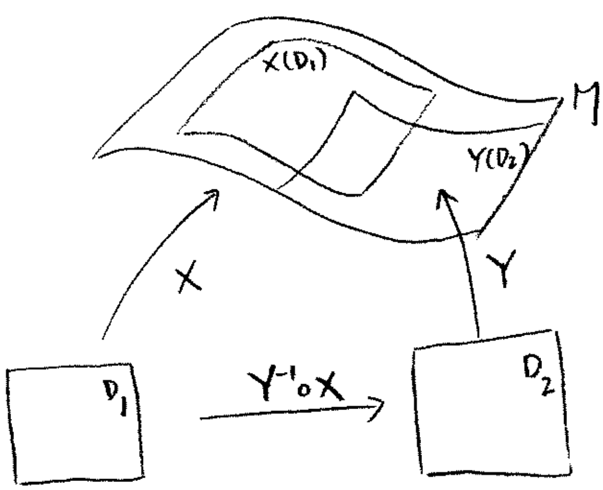
if $\beta : I \to \mathbb{R}^2$ is diff'ble then $X \circ \beta = \alpha$ is too. Your job is to show the converse. Hint: "fatten things up"

Define $Z(u,v,w) = X(u,v) + w\left(\dfrac{X_u \times X_v}{\|X_u \times X_v\|}\right)$



Use Math 3220 inverse fcn thm in a nbhds of $\beta(t_0)$ & $\alpha(t_0)$

(II)



If M is a (differentiable) surface (our def. page 3 of notes), and if X and Y are patches, then <u>show</u> $Y^{-1} \circ X : (X^{-1}(Y(D_2)) \to Y^{-1}(X(D_1))$ is differentiable.

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \cap \quad\quad\quad\quad\quad\quad \cap$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad D_1 \quad\quad\quad\quad\quad\quad D_2$

Hint: use a fattening idea like in I, to deal with the Y^{-1} differentiability question