Math 4530
Fri 25 Feb

Hand in this HW assignment Monday; no new hw this week.
(xerox it if you want copy to study)
On Monday I'll provide sol'ns to this assignment,
also to the 1st two assignments, which I will return. (graded if possible)

Exam Friday will cover chapters 1 – 3.3, i.e. thru the current HW.
conceptual in nature; basic computations too.

Recall normal curvature
$p \in M$
$v \in T_pM$, $|v| = 1$

$k_n(\vec v) = S(\vec v)\cdot \vec v \quad = \alpha''\cdot U$ for any curve $\alpha : I \to M$
$\alpha(t_0) = p$
$\alpha'(t_0) = \vec v$

($\text{so} = \kappa\, \vec N \cdot \vec U$ if α pbal,)
$\alpha''(t_0) = \kappa \vec N$

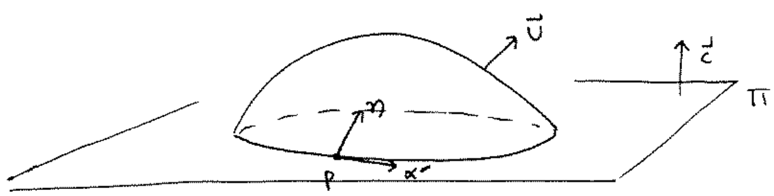
- How normal curvature depends on direction in T_pM,
  see page 3 Wed.

In particular, if $\vec v$ is a unit principal direction $(S(\vec v) = k_i \vec v \ , \ i = 1 \text{ or } 2)$
then $S(\vec v)\cdot \vec v = k_i$

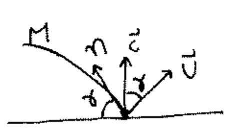
So if we can identify principal dirs we can compute principal curvatures as normal curvatures.

Theorem Let $\alpha \subset M \cap \pi$, π a plane with unit normal $\vec c$. Suppose $\gamma := \measuredangle\, \vec U, \vec c$ is constant, $0 < \gamma \le \pi/2$ along α.
γ = angle btwn M & π

Then α is a line of curvature!



α' into the paper:



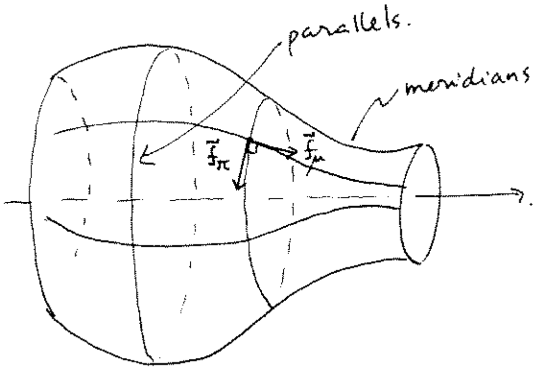
proof: $S(\alpha') = a\alpha' + b\eta$
along α, where $\eta \perp \alpha'$ is the (unit) conormal vector in T_pM to α'.

But $(U\circ\alpha)\cdot \vec c \equiv$ const
$\Rightarrow \underbrace{-(U\circ\alpha)'(t)}\cdot \vec c \equiv 0$
$(a\alpha' + b\eta)\cdot \vec c \equiv 0$
$\Rightarrow b\, \vec\eta \cdot \vec c \equiv 0 \Rightarrow b \equiv 0$ ∎

Example: Revisit Surfaces of revolution: The meridians lie on planes thru the axis of revolution and the surface is $\perp$ to these planes

$\Rightarrow$ meridians are lines of curvature

$\Rightarrow$ parallels are too (spectral thm!)



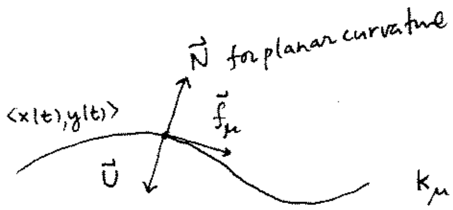
But, we can also deduce that the parallels are lines of curvature from page 1 theorem!

Thus, if at $P \in M$ we pick o.n. basis $\{\vec{f}_\mu, \vec{f}_\pi\}$ for T_pM

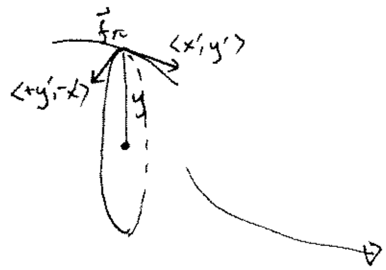
$\uparrow$ meridian dir $\nwarrow$ parallel dir.

it is an eigenbasis for S.

And, we can "see" the principal curvatures



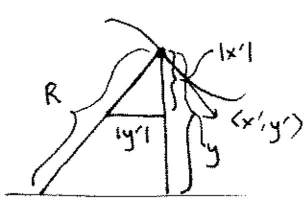
$$k_\mu = S(f_\mu)\cdot f_\mu = \kappa \underbrace{\vec{N}\cdot\vec{U}}_{-1}$$
$$= -\kappa_{planar}$$
$$= \frac{x''y' - x'y''}{(x'^2+y'^2)^{3/2}}$$



$$k_\pi = S(\vec{f}_\pi)\cdot\vec{f}_\pi$$
$$= \vec{\alpha}''\cdot\vec{U} \quad \text{for } \alpha \text{ pbal}$$
$$= \kappa\vec{N}\cdot\vec{U} \quad \leftarrow \text{for circle of radius } y$$
$$= \langle 0, -\tfrac{1}{y}\rangle \cdot \langle y', -x'\rangle \frac{1}{\sqrt{x'^2+y'^2}}$$
$$= \frac{x'}{y\sqrt{x'^2+y'^2}}$$

$$\frac{y}{R} = \frac{|x'|}{\sqrt{x'^2+y'^2}}$$

$$R = \frac{y\sqrt{x'^2+y'^2}}{|x'|}.$$



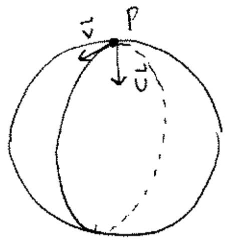
in fact the curvature $k_\pi = \frac{1}{R}$,

R = distance along $\vec{U}$ from P to axis

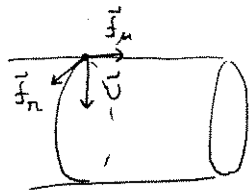
Early examples revisited.

sphere of radius R : Every great circle (actually every circle) is a line of curvature

If $\vec{U}$ = inner normal $\quad S(\vec{v})\cdot\vec{v} = \frac{1}{R} \quad$ ($\vec{v}$ unit).



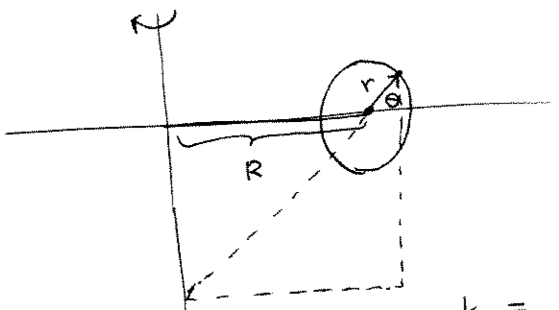
cylinder radius R

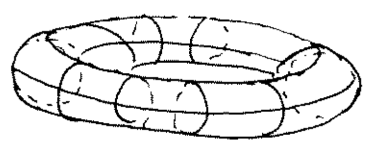


$S(\vec{f}_\mu) = 0$
$S(\vec{f}_\pi) = \frac{1}{R}\vec{f}_\pi \quad$ if $\vec{U}$ = inner normal.

torus of revolution

$\vec{U}$ = inner normal
$k_\mu = \frac{1}{r}$





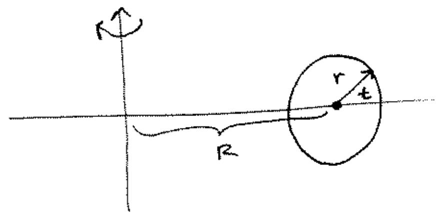
$$k_\pi = \frac{1}{\frac{R + r\cos\theta}{\cos\theta}} = \frac{\cos\theta}{R + r\cos\theta} \qquad \text{(even when } \cos\theta \overset{?}{<} 0\text{)}.$$

check : Last Friday's derivation of shape operator matrix for surface of revolution assumed rotation of curve in x-y plane about x-axis, but if we write $g(t)$ for dist along axis (rather than $x(t)$)
$h(t)$ for dist from axis (rather than $y(t)$).

Then what we labeled the axis doesn't matter, and

$$[S] = \begin{bmatrix} \underbrace{\dfrac{g''h' - g'h''}{(g'^2+h'^2)^{3/2}}}_{k_\mu} & 0 \\ 0 & \underbrace{\dfrac{1}{h}\dfrac{g'}{\sqrt{g'^2+h'^2}}}_{k_\pi} \end{bmatrix}$$

So for the torus,



$g(t) = r\sin t$ coord dist along axis dir.

$h(t) = R + r\cos t$ dist from axis

$$k_\mu = \frac{g''h' - g'h''}{(g'^2 + h'^2)^{3/2}} = \frac{(-r\sin)(-r\sin) - (r\cos)(-r\cos)}{r^3} = \frac{1}{r} \checkmark$$

$$k_\pi = \frac{1}{h}\frac{g'}{\sqrt{g'^2 + h'^2}} = \frac{1}{R + r\cos}\,\frac{+r\cos}{r} = \frac{\cos t}{R + r\cos t} \checkmark$$