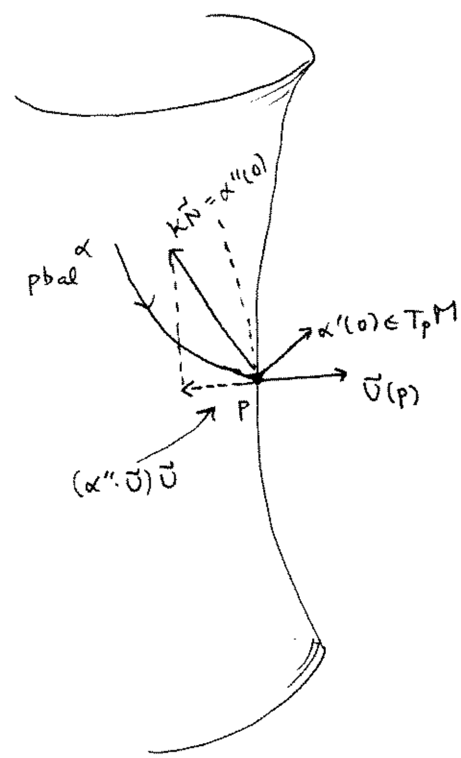
Math 4530
Wed 23 Feb

Recall from Friday

- the 1^{st} and 2^{nd} fundamental forms I, II
- their matrices, given a patch $X(u,v)$
  (the matrix of a quadratic form is not the same as the matrix of a linear transformation, in terms of what the entries mean!
- The relation between $[I]$, $[II]$, $[S]$.
- Work surfaces of revolution example.

Normal curvature



Let $\alpha : I \to M^2$ any regular curve
$\alpha(t_0) = P \in M$
Then the acceleration component $\perp$ to the surface, i.e. $\alpha''(t_0)\cdot\vec{U}$ is determined by the shape operator at P, and $\vec{v} = \alpha'(t_0)$

In fact:

$$\alpha' \cdot \vec{U} \equiv 0$$

$$\Rightarrow \quad \alpha'' \cdot \vec{U} + \alpha' \cdot (U\circ\alpha)' = 0$$

$$\alpha''\cdot\vec{U} = -\alpha'\cdot(U\circ\alpha)' = \alpha'\cdot S(\alpha') = II(\alpha',\alpha')$$

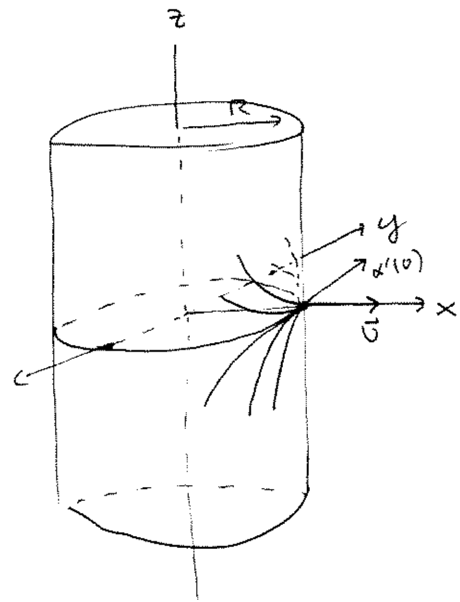
Def $P \in M$, $\vec{v} \in T_pM$, $|\vec{v}| = 1$
The normal curvature of M, at p, in dir $\vec{v}$

$$k_n(\vec{v}) := \vec{v}\cdot S(\vec{v})$$

By computation above, $k_n(\vec{v})$ represents the $\vec{U}$ component of the curvature for any curve passing thru P with unit tangent vector $\vec{v}$,

since $\alpha''\cdot\vec{U} = \kappa\vec{N}\cdot\vec{U}$ if α pbal
$\quad\quad\; \| $
$\;\alpha'\cdot S(\alpha')$

<u>Example</u> : cylinder of radius R



$P = (R,0,0)$, $U_p = \langle 1,0,0\rangle$

$\alpha(t) = \langle R\cos t, R\sin t, Ct^2\rangle$
@ $t=0$.

$\alpha'(0) = \langle 0, R, 0\rangle$

$\alpha''(0) = \langle -R, 0, 2C\rangle$

$\alpha''(0)\cdot\vec{U}_{\alpha(0)} = -R \quad \forall C.$

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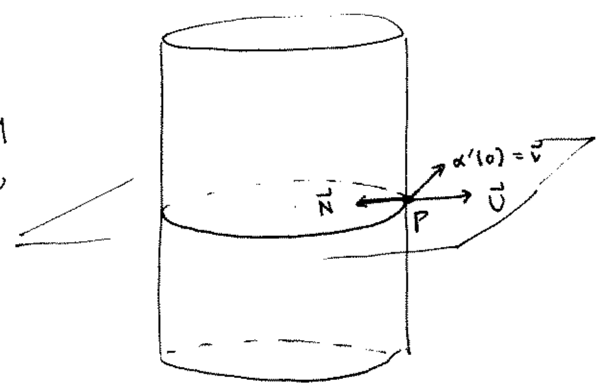
$S(\alpha'(0))\cdot\alpha'(0)$

Can use this to understand shape operator:

in this example, $\vec{v} = \langle 0,1,0\rangle = \dfrac{\alpha'(0)}{|\alpha'(0)|}$

$$S(\vec{v})\cdot\vec{v} = \frac{1}{|\alpha'(0)|^2}\, S(\alpha'(0))\cdot\alpha'(0) = \frac{1}{R^2}(-R) = -\frac{1}{R}.$$

<u>Important note</u> : If, for $p\in M$, $v\in T_pM$, $|\vec{v}|=1$, you choose the curve α which is the trace curve of the intersection between M & the plane thru p with directions $\vec{U}$ and $\vec{v}$, then $k_n(\vec{v}) = \pm\kappa$, depending on whether $\vec{U} = \pm\vec{N}$.



α in plane thru P
w dirs $\{\vec{v}, \vec{U}\}$

so $\vec{\alpha}\cdot\vec{c} \equiv 0$ $\quad (c = \vec{v}\times\vec{U}$ at P$)$

$\Rightarrow \alpha'\cdot\vec{c} \equiv 0$
$\alpha''\cdot\vec{c} \equiv 0$

$\Rightarrow \kappa\vec{N} \in$ plane, $\perp \vec{v}$

$\Rightarrow \kappa\vec{N} = \kappa(\pm\vec{U})$.

$\Rightarrow \kappa = \pm\,\alpha''(0)\cdot\vec{U}$
$= \pm\, k_n(\vec{v})$

How normal curvatures depend on direction:

Let $\{\vec{f}_1, \vec{f}_2\}$ be orthonormal eigenbasis for shape operator at P, $\quad S(\vec{f}_1) = k_1 \vec{f}_1 \qquad k_1 \le k_2$

$S(\vec{f}_2) = k_2 \vec{f}_2$

Let $\vec{v} = \cos\theta\, f_1 + \sin\theta\, f_2$

Then

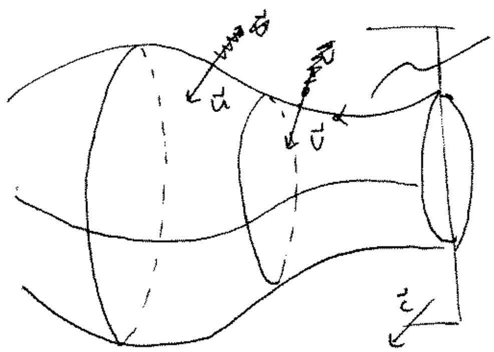
$$k_n(\vec{v}) = S(\vec{v}) \cdot \vec{v} = \underbrace{S(\cos\theta\, f_1 + \sin\theta\, f_2)}_{\cos\theta\, k_1 \vec{f}_1 \,+\, \sin\theta\, k_2 \vec{f}_2} \cdot (\cos\theta\, \vec{f}_1 + \sin\theta\, \vec{f}_2)$$

$$= k_1 \cos^2\theta + k_2 \sin^2\theta \quad ;$$

so $k_1 \le k_n(\vec{v}) \le k_2$

and $k_n(\vec{v})$ has average value $H = \frac{k_1 + k_2}{2}$.

Surfaces of revolution revisited



profile curves must be lines of curvature

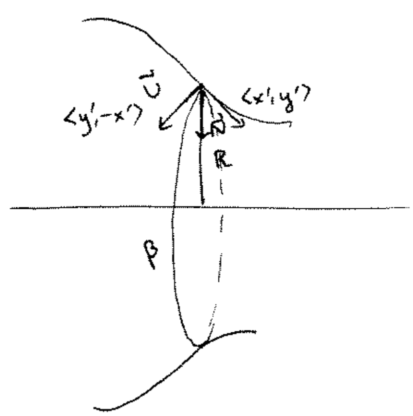
because curve α and surface normal stay in fixed plane. ($\perp$ to fixed dir. $\vec{c}$)

Thus $S(\alpha') = (U \circ \alpha)'$ is also $\perp \vec{c}$

$\Rightarrow S(\alpha') \parallel \alpha'$

$\Rightarrow k_1 = S(\alpha') \cdot \alpha' = \pm$ profile curve curvature depending on $\vec{U} = \pm \vec{N}$.

Since principal directions are $\perp$ to each other, latitude circles also must be lines of curvature.



$$k_2 = S(\beta') \cdot \beta'$$
$$= \beta'' \cdot \vec{U}$$
$$= \frac{1}{R} \vec{N} \cdot \vec{U}$$
$$= \frac{1}{R}(-x')$$

Compare with page 3 Fri