

Math 4530
Wed 2 Feb

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Multi-variable FTC

1st recall 1-variable version:

$$\int_a^b f(x) dx = F(b) - F(a), \text{ where } F'(x) = f(x)$$

$$\text{or } \int_a^b F'(x) dx = F(b) - F(a)$$

A good way to think of this:

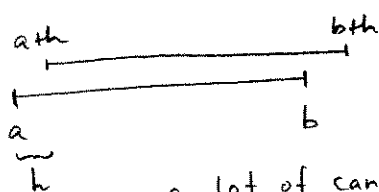
$$F'(x) \approx \frac{F(x+h) - F(x)}{h}$$

$$\begin{aligned} \text{so } \int_a^b F'(x) dx &\stackrel{(1)}{\approx} \int_a^b \frac{F(x+h) - F(x)}{h} dx \\ &= \frac{1}{h} \int_a^b F(x+h) dx - \frac{1}{h} \int_a^b F(x) dx \end{aligned}$$

$$\begin{aligned} z &= x+h \\ dz &= dx \end{aligned}$$

(translation of variables)

$$= \frac{1}{h} \int_{a+h}^{b+h} F(z) dz - \frac{1}{h} \int_a^b F(x) dx$$



a lot of cancellation!

$$= \frac{1}{h} \int_b^{b+h} F(x) dx - \frac{1}{h} \int_a^{a+h} F(x) dx$$

$$\stackrel{(2)}{\approx} F(b) - F(a)$$

In fact, as $h \rightarrow 0$ you can deduce

$$\int_a^b F'(x) = F(b) - F(a)$$

[For first $\approx$ you need e.g. $F'(x)$ continuous on $[a, b]$
For second $\approx$ you need e.g. $F(x)$ continuous at a & b]

Details:

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By MVT

$$\frac{F(x+h) - F(x)}{h} = F'(c_x), |x - c_x| < h$$

$$\begin{aligned} \text{so } \left| \int_a^b F'(x) - \left(\frac{F(x+h) - F(x)}{h} \right) dx \right| \\ \leq \int_a^b |F'(x) - F'(c_x)| dx, \end{aligned}$$

$$|x - c_x| < h$$

If F' unif cont on $[a, b]$

then $\forall \epsilon > 0 \exists \delta > 0$ s.t.

$$|x - c_x| < \delta$$

$$\Rightarrow |F'(x) - F'(c_x)| < \epsilon$$

in which case,

$$\leq \int_a^b \epsilon dx = \epsilon(b-a) \blacksquare$$

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$$\approx \left| \frac{1}{h} \int_b^{b+h} F(x) dx - F(b) \right|$$

$$= \left| \frac{1}{h} \int_b^{b+h} F(x) - F(b) dx \right|$$

F cont at $b \Rightarrow \forall \epsilon > 0 \exists \delta > 0$

$$\text{s.t. } |F(x) - F(b)| < \epsilon$$

when $|x - b| < \delta$

in which case

$$\leq \frac{1}{h} \int_b^{b+h} \epsilon dx = \epsilon \blacksquare$$

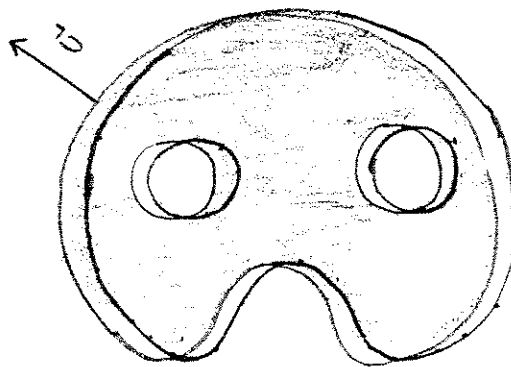
The same idea is behind $\mathbb{R}^n$ version
(although proof usually follows a technically simpler route)

Illustrate when $n=2$

Domain D (shaded), $\partial D = \text{bdry curve}$

$$D_h := \{z = x + h\vec{e}_1 \mid x \in D\}$$

is translation by amt h in $\vec{e}_1$ direction



$\vec{U} = \text{unit exterior normal}$

$$F: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\iint_D \frac{\partial F}{\partial x} dA \stackrel{\textcircled{1}}{\approx} \iint_D \frac{F(\vec{x} + h\vec{e}_1) - F(\vec{x})}{h} dA$$

$$= \frac{1}{h} \iint_D F(\vec{x} + h\vec{e}_1) dA - \frac{1}{h} \iint_D F(\vec{x}) dA$$

$$\vec{z} = \vec{x} + h\vec{e}_1$$

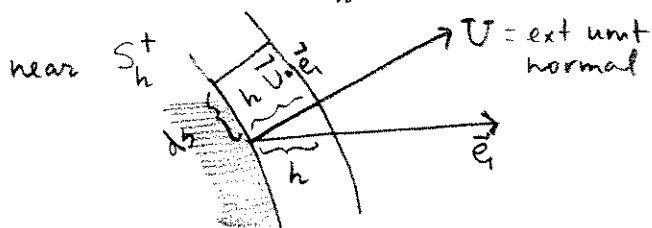
$$dA_z = dA_x$$

(translation of domains)

$$= \frac{1}{h} \iint_{D_h} F(\vec{z}) dA_z - \frac{1}{h} \iint_D F(\vec{x}) dA_x$$

lots of cancellation in the middle

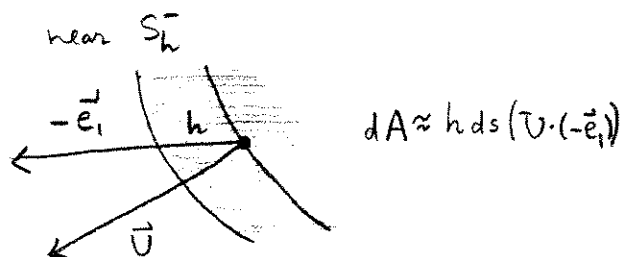
$$* = \frac{1}{h} \iint_{S_h^+} F(x) dA - \frac{1}{h} \iint_{S_h^-} F(x) dA$$



$$dA \approx ds h (\vec{U} \cdot \vec{e}_1)$$

$$S_h^+ = D_h \setminus D = \text{unshaded strip(s)}$$

$$S_h^- = D \setminus D_h = \text{shaded strip(s)}$$



$$dA \approx h ds (\vec{U} \cdot (-\vec{e}_1))$$

$$\text{so } * \stackrel{\textcircled{2}}{\approx} \int_{\partial D} F \vec{e}_1 \cdot \vec{U} ds$$

(h's cancel, also signs in 2nd term)

In the limit

$$\boxed{\iint_D \frac{\partial F}{\partial x} dA = \int_{\partial D} F \vec{e}_1 \cdot \vec{U} ds}$$

$\textcircled{1}$ and $\textcircled{2}$ are messy to prove, so rigorous proof is exposited slightly differently in general (page 4), but with this idea hiding behind it.

$\mathbb{R}^1$ $\int_a^b F'(x) dx = F(b) - F(a)$	$\mathbb{R}^n$ $\int_{D_n} \frac{\partial F}{\partial x_i} dVol_n = \int_{\partial D_n} F(\vec{x}) \cdot \vec{U} dVol_{n-1}$
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SAME!

Special forms: Divergence Theorem

$$\text{div} \begin{bmatrix} P \\ Q \end{bmatrix} := P_x + Q_y$$

$$\begin{aligned} \iint_D P_x dA &= \int_{\partial D} P \vec{e}_1 \cdot \vec{U} ds \\ + \iint_D Q_y dA &= \int_{\partial D} Q \vec{e}_2 \cdot \vec{U} ds \\ \hline \iint_D \text{div} \begin{bmatrix} P \\ Q \end{bmatrix} dA &= \int_{\partial D} \begin{bmatrix} P \\ Q \end{bmatrix} \cdot \vec{U} ds \end{aligned}$$

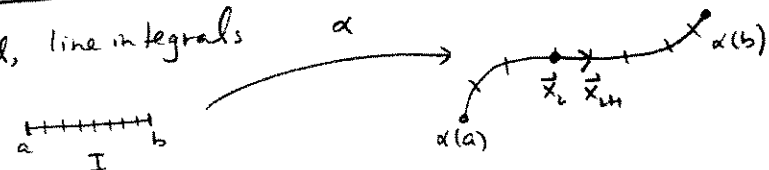
$\vec{F}(\vec{x})$ a vector field

in $\mathbb{R}^n$

$$\int_{D_n} \text{div} \vec{F} dVol_n = \int_{\partial D_n} \vec{F} \cdot \vec{U} dVol_{n-1}$$

Green's thm & Stoke's Thm in $\mathbb{R}^2$

1st recall, line integrals



$$\begin{aligned} \int_{\alpha} \vec{F} \cdot d\vec{x} &:= \lim_{\|P\| \rightarrow 0} \sum \vec{F}(\vec{x}_i) \cdot (\vec{x}_{i+1} - \vec{x}_i) \\ &= \int_I \vec{F}(\alpha(t)) \cdot \alpha'(t) dt \end{aligned}$$



$D \subset \mathbb{R}^2$

Green:
$$\iint_D Q_x - P_y dA = \oint_{\partial D} P dx + Q dy$$

Stokes
 $M^2 \subset \mathbb{R}^3$

$$\iint_M (\nabla \times \vec{F}) \cdot \vec{U} dA = \int_{\partial M} \vec{F} \cdot d\vec{x}$$

 (parametric version of Green's)

depends on direction image curve is traversed, but not on particular parameterization. Line integrals can compute work

by div thm.

Now parameterize ∂D by arclength

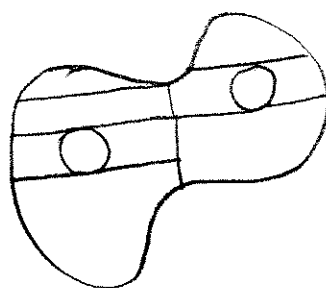
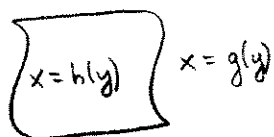
$$\begin{aligned} \text{RHS} &= \int_{\partial D} \begin{bmatrix} Q \\ -P \end{bmatrix} \cdot \begin{bmatrix} \cos \\ \sin \end{bmatrix} ds \\ &= \int_{\partial D} \begin{bmatrix} P \\ Q \end{bmatrix} \cdot \begin{bmatrix} -\sin \\ \cos \end{bmatrix} ds \end{aligned}$$

$\begin{bmatrix} P \\ Q \end{bmatrix} \cdot \begin{bmatrix} -\sin \\ \cos \end{bmatrix} = P_x + Q_y$

The way towards a rigorous proof of n-variable FTC
(illustrated for $n=2$)

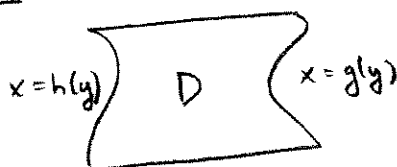
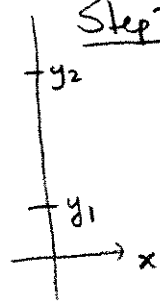
to show
$$\iint_D \frac{\partial F}{\partial x} dA = \int_{\partial D} F \vec{e}_1 \cdot \vec{U} ds$$

Step 1 Decompose D into pieces of type



if can show for basic piece,
add over all pieces, notice interior edge contributions from adjacent
pieces cancel, so get thm.

Step 2



$$\iint_D \frac{\partial F}{\partial x} dA = \iint_D \frac{\partial F}{\partial x} dx dy = \int_{y_1}^{y_2} \left(\int_{h(y)}^{g(y)} \frac{\partial F}{\partial x} dx \right) dy$$

Fubini

$$= \int_{y_1}^{y_2} F(y, g(y)) - F(y, h(y)) dy$$

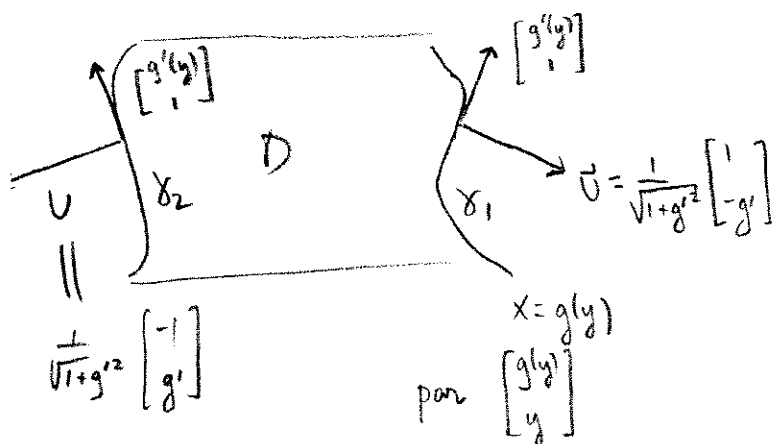
1-var
FTC

$$= \int_{\gamma_1} F dy - \int_{\gamma_2} F dy$$

$$= \int_{\gamma_1} F \vec{e}_1 \cdot \vec{U} ds - \int_{\gamma_2} F \vec{e}_1 \cdot \vec{U} ds$$

but on γ_1 , $ds = \sqrt{1+g'^2} dy$; $dy = \frac{1}{\sqrt{1+g'^2}} ds$

$$= \int_{\partial D} F(\vec{e}_1 \cdot \vec{U}) ds$$



Let's try to write procedures for the T.N.B frame. We may assume the user has already loaded the curtor procedures.

```
> TNB:=proc(alpha)
  local alphap, alphapp, alphappp, num1, den1,
    num2, den2, num3, den3,
    T11, T12, T13, T1, N11, N12, N13, N1,
    B11, B12, B13, B1;
  alphap:=diff(alpha, t);
  alphapp:=diff(alphap, t);
  alphappp:=diff(alphapp, t);
  den1:=simplify(nrm(alphap), trig, sqrt, symbolic);
  T11:=simplify(alphap[1]/den1);
  T12:=simplify(alphap[2]/den1);
  T13:=simplify(alphap[3]/den1);
  T1:=[T11, T12, T13];
  num2:=simplify(xp(alphap, alphapp), trig, sqrt, symbolic);
  den2:=simplify(nrm(num2), trig, sqrt, symbolic);
  B11:=simplify(num2[1]/den2);
  B12:=simplify(num2[2]/den2);
  B13:=simplify(num2[3]/den2);
  B1:=[B11, B12, B13];
  N1:=simplify(xp(T1, B1), trig, sqrt, symbolic);
```

I got a better procedure by simplifying the components of T, N, B separately

oops, my cat ate the HW!

but here are my answers

```
> assume(a>0); assume(b>0);
hel:=[a*cos(t), a*sin(t), b*t];
TNB(hel);
```

$$hel := [a \cos(t), a \sin(t), b t]$$

$$T = \left[-\frac{a \sin(t)}{\sqrt{a^2 + b^2}}, \frac{a \cos(t)}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \right], N = [-\cos(t), -\sin(t), 0],$$

$$B = \left[\frac{b \sin(t)}{\sqrt{a^2 + b^2}}, -\frac{b \cos(t)}{\sqrt{a^2 + b^2}}, \frac{a}{\sqrt{a^2 + b^2}} \right]$$

```
> Bcurve:=[exp(t)*cos(t), exp(t)*sin(t), exp(t)];
Bcurve:=[e^t cos(t), e^t sin(t), e^t]
```

```
> TNB(Bcurve);
```

$$T = \left[-\frac{1}{3}(\sin(t) - \cos(t))\sqrt{3}, \frac{1}{3}(\sin(t) + \cos(t))\sqrt{3}, \frac{\sqrt{3}}{3} \right],$$

$$N = \left[-\frac{1}{2}(\sin(t) + \cos(t))\sqrt{2}, -\frac{1}{2}\sqrt{2}(\sin(t) - \cos(t)), 0 \right],$$

$$B = \left[\frac{1}{6}(\sin(t) - \cos(t))\sqrt{6}, -\frac{1}{6}(\sin(t) + \cos(t))\sqrt{6}, \frac{\sqrt{6}}{3} \right]$$