Math 4930

Friday 18 Feb.

- Gauss curvature as area expansion factor for the normal map $U : M^2 \to S^2$

(page 3 Wed.)

Let's get computing! (See new HW).

- How to find the matrix of S, wrt $\{\vec{X}_u, \vec{X}_v\}$
- Formulas for H, K.

Let $[S] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

This means
$$S(X_u) = a_{11} X_u + a_{21} X_v$$
$$S(X_v) = a_{12} X_u + a_{22} X_v$$

i.e. in $\mathbb{R}^3$ coords:
$$\underbrace{\begin{bmatrix} S(X_u) \,\vdots\, S(X_v) \end{bmatrix}}_{3\times 2} = \underbrace{\begin{bmatrix} X_u \,\vdots\, X_v \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{[S]}$$

Thus
$$\begin{bmatrix} X_u^T \\ X_v^T \end{bmatrix} \begin{bmatrix} S(X_u) \,\vdots\, S(X_v) \end{bmatrix} = \begin{bmatrix} X_u^T \\ X_v^T \end{bmatrix} \begin{bmatrix} X_u \,\vdots\, X_v \end{bmatrix} [S]$$

text: $\begin{bmatrix} l & m \\ m & n \end{bmatrix} = [II]$

$$\underbrace{\begin{bmatrix} X_u \cdot S(X_u) & X_u \cdot S(X_v) \\ X_v \cdot S(X_u) & X_v \cdot S(X_v) \end{bmatrix}} = \underbrace{\begin{bmatrix} X_u \cdot X_u & X_u \cdot X_v \\ X_v \cdot X_u & X_v \cdot X_v \end{bmatrix}} [S]$$

this is called the matrix of the 2^{nd} fdl form, since you can use it to compute $\vec{w} \cdot S(\vec{z})$ for $\vec{w}, \vec{z} \in T_pM$.

this is called the matrix of the first fundamental form. (because the dot product of $\mathbb{R}^3$, restricted to T_pM, is called the first fdl form of T_pM, and
$$(aX_u + bX_v) \cdot (cX_u + dX_v) = [a, b] \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

and in our text its entries are $\begin{bmatrix} E & F \\ F & G \end{bmatrix} = [I]$

This matrix is invertible by Lagrange's identity (HW 3.1.5 p.117)
$$\begin{vmatrix} \vec{v} \cdot \vec{a} & \vec{v} \cdot \vec{b} \\ \vec{w} \cdot \vec{a} & \vec{w} \cdot \vec{b} \end{vmatrix} = (\vec{v} \times \vec{w}) \cdot (\vec{a} \times \vec{b})$$

Another way to write the entries in this matrix:
$$\begin{bmatrix} X_{uu} \cdot U & X_{uv} \cdot U \\ X_{uv} \cdot U & X_{vv} \cdot U \end{bmatrix}$$

because
$$X_u \cdot U \equiv 0 \Rightarrow X_{uu} \cdot U + X_u \cdot U_u = 0 \Rightarrow X_{uu} \cdot U = X_u \cdot S(X_u)$$
$$X_{uv} \cdot U + X_u \cdot U_v = 0 \Rightarrow X_{uv} \cdot U = X_u \cdot S(X_v)$$

etc.

Thus, $[S]_{\{X_u, X_v\}} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} \ell & m \\ m & n \end{bmatrix} = \frac{1}{EG-F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} \ell & m \\ m & n \end{bmatrix}$

$$K = \det S = \frac{\begin{vmatrix} \ell & m \\ m & n \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}}$$

$$H = \tfrac{1}{2} \operatorname{trace} S = \frac{1}{2} \, \frac{1}{EG-F^2} \left(G\ell - 2Fm + En \right)$$

Examples : graph above tangent plane at $p \in M$

$X(u,v) = p + u\vec{f}_1 + v\vec{f}_2 + k(u,v)\vec{f}_3$ $\qquad$ $\{\vec{f}_1, \vec{f}_2, \vec{f}_3\}$ orthonormal

$k_u = k_v = 0$ at $(0,0)$.

at $(u,v) = (0,0)$

$X_u = \vec{f}_1$

$X_v = \vec{f}_2$

$X_{uu} = k_{uu}\vec{f}_3$

$X_{uv} = k_{uv}\vec{f}_3$

$X_{vv} = k_{vv}\vec{f}_3$

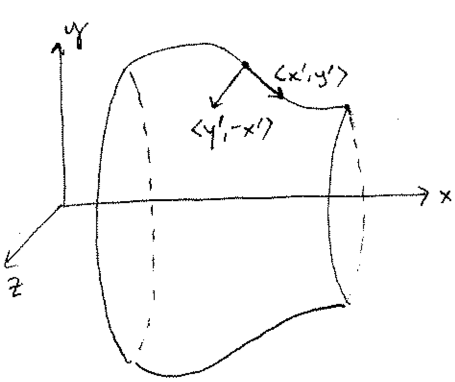
$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \ell & m \\ m & n \end{bmatrix} = \begin{bmatrix} k_{uu} & k_{uv} \\ k_{vu} & k_{vv} \end{bmatrix}$$

$$[S] = \begin{bmatrix} k_{uu} & k_{uv} \\ k_{vu} & k_{vv} \end{bmatrix}$$

as before. , also K, H

In HW you generalize these formulas
to arbitrary graph $z = k(x,y)$ (not above tangent plane)
3.2.26 p128
and apply result to quadric surfaces, 3.2.13, 3.2.14

Example : general surface of revolution (you had a torus example in current HW).



curve $\vec{r}(t) = \langle x(t), y(t) \rangle$, $y > 0$

rotate abt x-axis

$X(u,v) = \langle x(u), y(u)\cos v, y(u)\sin v \rangle$

$X_u = \langle x', y'\cos, y'\sin \rangle$

$X_v = \langle 0, -y\sin, y\cos \rangle$

$$[I] = \begin{bmatrix} (x')^2 + (y')^2 & 0 \\ 0 & y^2 \end{bmatrix}$$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x' & y'\cos & y'\sin \\ 0 & -y\sin & y\cos \end{vmatrix} = \langle yy', -yx'\cos, -yx'\sin \rangle = y\langle y', -x'\cos, -x'\sin \rangle$$

$$U = \frac{1}{\sqrt{x'^2 + y'^2}} \langle y', -x'\cos, -x'\sin \rangle$$

$$X_{uu} = \langle x'', y''\cos, y''\sin \rangle$$
$$X_{uv} = \langle 0, -y'\sin, y'\cos \rangle$$
$$X_{vv} = \langle 0, -y\cos, -y\sin \rangle$$
$$\vec{U} = \frac{1}{x'^2+y'^2} \langle y', -x'\cos, -x'\sin \rangle$$

$$[\mathrm{II}] = \frac{1}{\sqrt{x'^2+y'^2}} \begin{bmatrix} x''y'-x'y'' & 0 \\ 0 & yx' \end{bmatrix}$$

$$[S] = [\mathrm{I}]^{-1}[\mathrm{II}]$$
$$= \frac{1}{y^2(x'^2+y'^2)} \begin{bmatrix} y^2 & 0 \\ 0 & x'^2+y'^2 \end{bmatrix} \frac{1}{\sqrt{x'^2+y'^2}} \begin{bmatrix} x''y'-x'y'' & 0 \\ 0 & yx' \end{bmatrix}$$
$$= \begin{bmatrix} \frac{x''y'-x'y''}{(x'^2+y'^2)^{3/2}} & 0 \\ 0 & \frac{1}{y}\frac{x'}{\sqrt{x'^2+y'^2}} \end{bmatrix}$$

- latitudes and longitudes are <u>lines of curvature</u>, i.e. tangent to eigendirections of S (principal directions)
- the eigenvalue in the X_u direction is the signed curvature of the profile curve.

(What about the other eigenvalue?)

HW due next Friday 2/25 !!

Chapter 2:
2.15
3.4, 3.9, 3.10, 3.11 (assigned last week)
4.4, 4.6, 4.7

Chapter 3:
3.1.5, 3.1.6, 3.10, 3.11
3.2.4 (some chain rule practice)
3.2.6
3.2.26 do this before
3.2.13
3.2.14
3.2.15
3.3.6, 3.3.8