Math 4530
Wed 16 Feb.

Reminder: HW due <u>next</u> week!
(more to be added)
but can do problem session tomorrow.

<u>Recall from ~~Mon~~day.</u>

- If you parameterize surface M near $P \in M$ "above" T_pM (i.e. as a graph

$$Z(u,v) = P + u\vec{f}_1 + v\vec{f}_2 + g(u,v)\,\vec{f}_3$$

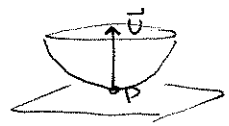
where $\{\vec{f}_1, \vec{f}_2\}$ o.n. basis for T_pM, $\vec{f}_3 = \vec{f}_1 \times \vec{f}_2$

Then $[S]_{\{\vec{f}_1,\vec{f}_2\}} = \begin{bmatrix} g_{uu} & g_{uv} \\ g_{vu} & g_{vv} \end{bmatrix}$ at P

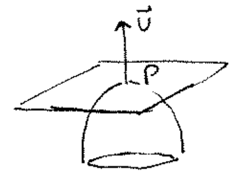
- If you choose o.n. eigenbasis $\{\vec{v}_1, \vec{v}_2\}$ (by rotating $\{\vec{f}_1, \vec{f}_2\}$), for T_pM,

$$[S]_{\{\vec{v}_1,\vec{v}_2\}} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$

evals k_1, k_2 "principal curvatures"



$k_1, k_2 > 0$
$K > 0$
$H > 0$



$k_1, k_2 < 0$
$K > 0$
$H < 0$

$k_1 > 0$ $K < 0$
$k_2 < 0$ H ?

also cases of $k_1 = 0$
$k_2 \neq 0$.
$K = 0$
H ?

- Discuss on page 3 Monday

$\det(S) := K$, Gauss curvature
$\frac{1}{2}\,\text{trace}(S) := H$ Mean curvature

- these are ~~propert~~ quantities of the linear map, not the particular matrix representing it
- Knowing K, H is equivalent to knowing k_1, k_2

see page 3 explanation, but here's a better one:

Let $[S]_{\mathcal{B}} = B$, $[S]_{\tilde{\mathcal{B}}} = \tilde{B}$ then $\tilde{B} = A^{-1}BA$ where A is a change of basis matrix.

then the characteristic polynomial

$$\begin{aligned}\det(B - \lambda I) &= \det(A^{-1})\det(B - \lambda I)\det A \\ &= \det\left(A^{-1}(B - \lambda I)A\right) \\ &= \det(\tilde{B} - \lambda I)\end{aligned}$$

is the same for B & $\tilde{B}$.

But look at its coeffs!

$$\det(B - \lambda I) = \begin{vmatrix} b_{11}-\lambda & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22}-\lambda & & \\ \vdots & & \ddots & \\ b_{n1} & & & b_{nn}-\lambda \end{vmatrix} = (-1)^n\lambda^n + (-1)^{n-1}\lambda^{n-1}\,\text{trace}(B) + \cdots + \lambda^0 \det B$$

Thus $\boxed{\begin{cases} H = \frac{1}{2}(k_1+k_2) \\ K = k_1 k_2 \end{cases}}$

$\cancel{\det(\lambda I - S)}$

And, $\det(S-\lambda I) = \lambda^2 - \lambda \operatorname{trace}(S) + \det(S)$

$= \lambda^2 - 2H\lambda + K$

so k_1, k_2 are roots of this eqtn

$$\boxed{k_i = H \pm \sqrt{H^2 - K}}$$

Example : From Monday

$[S] = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ $H = 2$, $K = -5$

$k_i = 2 \pm \sqrt{4+5}$
$= -1, 5$

$[S] = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$ $H = \frac{5-1}{2} = 2$, $K = -5$, $k_i = -1, 5$.

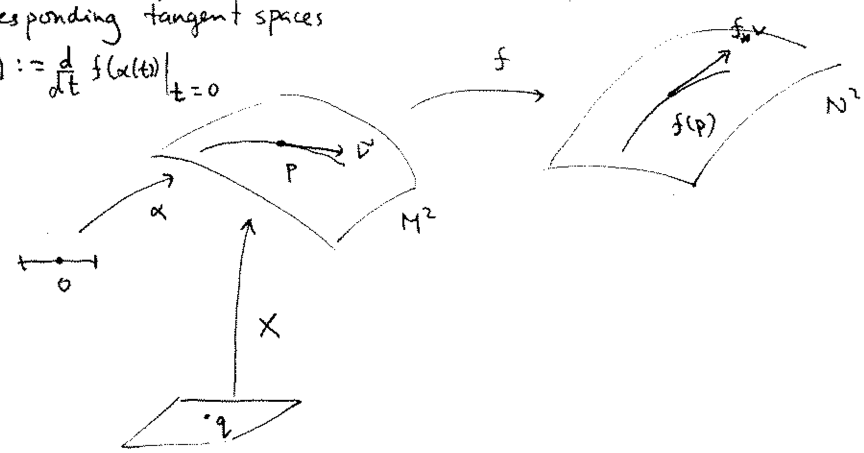
An interpretation of Gauss curvature

We can consider the unit normal U as a map from M^2 to the unit sphere S^2. The book writes $G = U$, for Gauss map

In general, if M^2, N^2 are surfaces, $f: M^2 \to N^2$ then the differential map $f_*: T_pM \to T_{f(p)}N$ is the linear map between corresponding tangent spaces

$\alpha(0) = p$
$\alpha'(0) = \vec{v} \in T_pM$

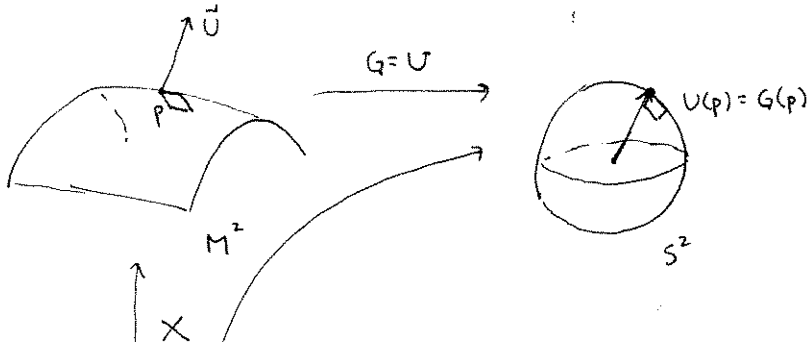
$f_*(\alpha'(0)) := \frac{d}{dt} f(\alpha(t))\Big|_{t=0}$

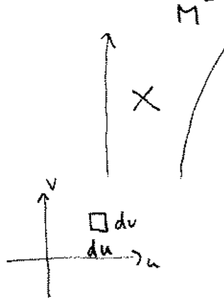


In local chart

$$f_*(a\vec{X}_u + b\vec{X}_v) = \frac{d}{dt}\left(f(q + t(a,b))\right)\Big|_{t=0}$$
$$= (f \circ X)_u a + (f \circ X)_v b$$

So $G_*(X_u) = U_u$
$G_*(X_v) = U_v$





on M^2,

$$dA = |\vec{X}_u \times \vec{X}_v|\, dv$$

on S^2

$$dA = |\vec{U}_u \times \vec{U}_v|\, du\, dv$$

relate to shape operator

$$S(X_u) = -U_u = a_{11} X_u + a_{21} X_v$$
$$S(X_v) = -U_v = a_{12} X_u + a_{22} X_v$$

$$[S]_{\{X_u, X_v\}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

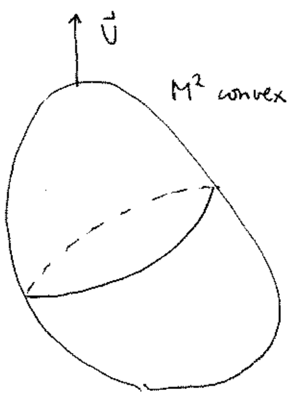
$$|U_u \times U_v| = |(a_{11} X_u + a_{21} X_v) \times (a_{12} X_u + a_{22} X_v)|$$
$$= \det(S)$$

$$\vec{U}_u \times \vec{U}_v = (a_{11} X_u + a_{21} X_v) \times (a_{12} X_u + a_{22} X_v)$$
$$= K(X_u \times X_v)$$

so $K > 0 \Rightarrow K = \dfrac{dA_{S^2}}{dA_{M^2}}$ i.e. $\displaystyle\iint_{M^2} K\, dA = \iint_{S^2} dA$ (possible overlap)

so $K < 0 \Rightarrow \displaystyle\iint_{M^2} K\, dA = -\iint_{S^2} dA$

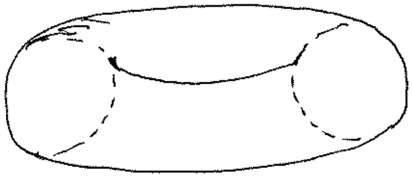
<u>example</u>



M^2 convex, i.e.
$U : M^2 \to S^2$
1-1 & onto

$$\iint_{M^2} K\, dA = \iint_{S^2} dA = 4\pi \;!$$

<u>example</u> What is $\displaystyle\iint_{M^2} K\, dA$ for this torus? of revolution



These are special cases of Gauss-Bonnet thm mentioned day 1 of class!