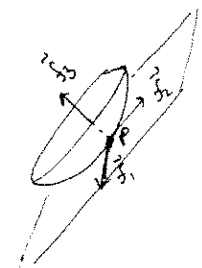
Math 4530
Monday 2/14

<u>Example</u>

Compute and interpret the shape operator at $P=(0,0,0)$, for the Monge patch

$$\vec{X}(x,y) = x\vec{e}_1 + y\vec{e}_2 + (x^2+y^2+3xy)\vec{e}_3$$

This is a special case of graphing a surface locally above its tangent plane at $P \in M^2$, which can be done for any point $\longrightarrow$ $Z(x,y) = P + x\vec{f}_1 + y\vec{f}_2 + g(x,y)\vec{f}_3$, $\nabla g(0,0)=\vec{0}$



$T_pM = \text{span}\{\vec{f}_1, \vec{f}_2\}$

$\vec{f}_3 = \vec{f}_1 \times \vec{f}_2 = \vec{U}(P)$

($\{\vec{f}_1, \vec{f}_2, \vec{f}_3\}$ orthonormal)

$$\vec{Z}_x = \begin{bmatrix}1\\0\\g_x\end{bmatrix} \qquad \vec{Z}_y = \begin{bmatrix}0\\1\\g_y\end{bmatrix}$$

(expressed in coords for the $\{\vec{f}_1, \vec{f}_2, \vec{f}_3\}$ basis)

$$\vec{U} = \begin{bmatrix}-g_x\\-g_y\\1\end{bmatrix}\frac{1}{\sqrt{1+g_x^2+g_y^2}}$$

$$\vec{X}_x = \begin{bmatrix}1\\0\\2x+3y\end{bmatrix}$$

$\leftarrow$ using coords wrt $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$: $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

$$\vec{X}_y = \begin{bmatrix}0\\1\\2y+3x\end{bmatrix}$$

$$\vec{U} = \begin{bmatrix}-2x-3y\\-2y-3x\\1\end{bmatrix}\frac{1}{\sqrt{1+Q(x,y)}}$$

$Q(x,y)$ quadratic in x,y.

at $P=(0,0,0)$, $\vec{X}_x = \vec{e}_1$, $\vec{X}_y = \vec{e}_2$

$$S(\vec{X}_x) = -\nabla_{\vec{X}_x}U = -\vec{U}_x$$

$$\text{at } P = \begin{bmatrix}+2\\+3\\0\end{bmatrix} = 2\vec{e}_1 + 3\vec{e}_2$$

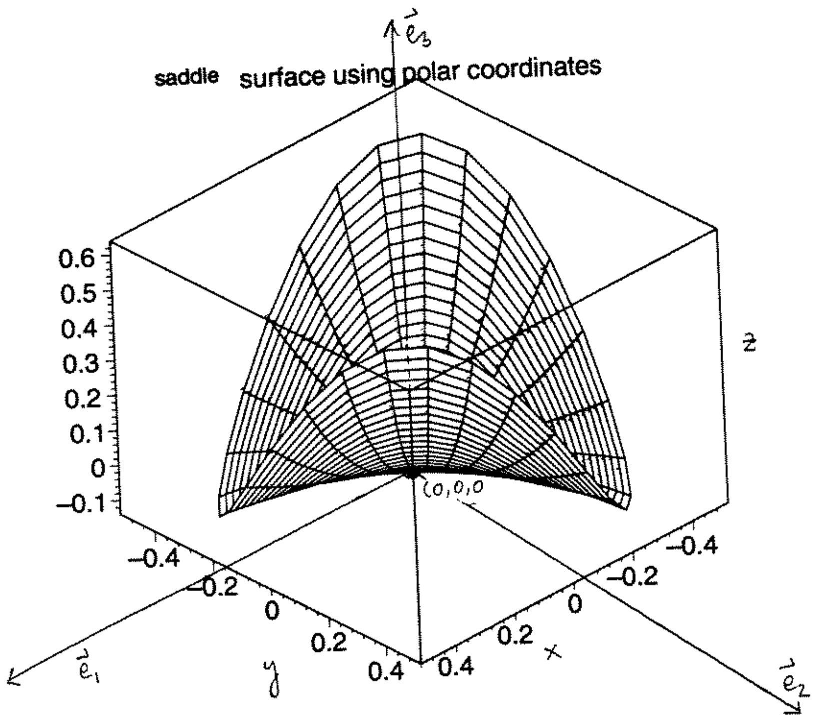
$$S(\vec{X}_y) = -\vec{U}_y = \begin{bmatrix}3\\2\\0\end{bmatrix} = 3\vec{e}_1 + 2\vec{e}_2$$

so at P,

$$[S]_{\{\vec{e}_1,\vec{e}_2\}} = \begin{bmatrix}2 & 3\\3 & 2\end{bmatrix}$$

$\uparrow [S(\vec{e}_1)]_{\{\vec{e}_1,\vec{e}_2\}}$ $\quad \uparrow [S(\vec{e}_2)]_{\{\vec{e}_1,\vec{e}_2\}}$

> Notice this is the Hessian matrix of 2^{nd} partials of x^2+y^2+3xy, evaluated at $(0,0)$



$z = x^2+y^2+3xy$

unit eigenvectors of S are called the <u>principal directions</u>

eigenvalues k_1, k_2 are <u>principal curvatures</u>

Find them!

$$S\vec{v} = \lambda \vec{v}$$

$$\Leftrightarrow [S\vec{v}]_{\{\vec{e}_1,\vec{e}_2\}} = \lambda [\vec{v}]_{\{\vec{e}_1,\vec{e}_2\}}$$

$$\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \lambda \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$0 = \begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} = (\lambda-2)^2 - 9 = (\lambda-2-3)(\lambda-2+3) = (\lambda-5)(\lambda+1)$$

k_1: $\lambda = 5$

$$\left[\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

k_2: $\lambda = -1$

$$\left[\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

New basis for T_pM:

$$\mathcal{B} = \left\{ \tfrac{1}{\sqrt{2}}(\vec{e}_1+\vec{e}_2),\ \tfrac{1}{\sqrt{2}}(-\vec{e}_1+\vec{e}_2) \right\} = \{\vec{w}_1, \vec{w}_2\}$$

$$S(\vec{w}_1) = 5\vec{w}_1$$
$$S(\vec{w}_2) = -\vec{w}_2$$

Can you see this in the page 1 picture?

general graph above tangent plane continued

$$Z_x(0,0) = \vec{f}_1$$
$$Z_y(0,0) = \vec{f}_2$$

So $S(\vec{f}_1) = -\nabla_{\vec{f}_1}\vec{U} = -\vec{U}_x$ at P

$$= \begin{bmatrix} +g_{xx}(\vec{0}) \\ +g_{yx}(\vec{0}) \\ 0 \end{bmatrix} = g_{xx}\vec{f}_1 + g_{yx}\vec{f}_2$$

$$S(f_2) = -\vec{U}_y \text{ at } P, = \begin{bmatrix} g_{xy} \\ g_{yy} \end{bmatrix}$$

at P,

$$[S] = \begin{bmatrix} g_{xx}(\vec{0}) & g_{xy}(\vec{0}) \\ g_{yx}(\vec{0}) & g_{yy}(\vec{0}) \end{bmatrix}$$

matrix wrt $\{\vec{f}_1, \vec{f}_2\}$ basis

<u>Remark</u>: This is another proof that the shape operator is self adjoint, since its matrix with respect to the orthonormal basis $\{\vec{f}_1, \vec{f}_2\}$ is symmetric.

<u>Thus</u>, as at left example, spectral theorem guarantees an ortho-normal eigenbasis for the shape operator

In fact, if we change coordinates to the $\{\vec{w}_1, \vec{w}_2\}$ basis

$$x\vec{e}_1 + y\vec{e}_2 = r\vec{w}_1 + s\vec{w}_2$$

$$\Rightarrow x = (r\vec{w}_1 + s\vec{w}_2)\cdot\vec{e}_1 = \tfrac{1}{\sqrt{2}}(r-s)$$
$$y = (r\vec{w}_1 + s\vec{w}_2)\cdot\vec{e}_2 = \tfrac{1}{\sqrt{2}}(r+s)$$

$$x^2 + y^2 + 3xy = \tfrac{1}{2}(r-s)^2 + \tfrac{1}{2}(r+s)^2 + \tfrac{3}{2}(r-s)(r+s) = \tfrac{5}{2}r^2 - \tfrac{1}{2}s^2$$

So better param is

$$\tilde{X}(r,s) = r\vec{w}_1 + s\vec{w}_2 + \left(\tfrac{5}{2}r^2 - \tfrac{1}{2}s^2\right)\vec{e}_3$$

(We <u>knew</u> this result based on shape computations on left, and general Hessian result on right)

Hessian discussion cont'd.

Recall for a fcn of 2 variables, the Hessian matrix is used to measure 2^{nd} directional derivatives at a point

If point is $(0,0)$, fcn is $g(x,y)$, direction is $\langle a,b\rangle$, then

$$\frac{d^2}{dt^2}\, g(ta,tb)\Big|_{t=0} = \Big(g_x(\ )a + g_y(\ )b\Big)_t\Big|_{t=0} = g_{xx}a^2 + g_{xy}ab + g_{yx}ab + g_{yy}b^2$$
$$= [a\ b]\begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy}\end{bmatrix}\begin{bmatrix} a \\ b\end{bmatrix}$$

If we use the rotated basis $\{\vec{w}_1, \vec{w}_2\}$ in page 1→2 example, this becomes (in the r-s variables)

$$\frac{d^2}{dt^2}\, g(t\cos\theta\,\vec{w}_1 + t\sin\theta\,\vec{w}_2)\Big|_{t=0} = [\cos\theta, \sin\theta]\begin{bmatrix} 5 & 0 \\ 0 & -1\end{bmatrix}\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}$$

$$= 5\cos^2\theta - \sin^2\theta$$

so these pure 2^{nd} directional derivatives are between -1 and 5, as the page 1 picture illustrated.

Gauss and Mean Curvature

If $S: V \to V$ is linear,
then $\det(S)$, $\text{trace}(S)$ ← (sum of diagonal elements)
are well defined, i.e. you can use the matrix of S with respect to any basis to compute them

(since $\det(ASA^{-1}) = (\det A)(\det S)\frac{1}{\det A} = \det S$

and since $\text{trace}(AB) = \sum_i [AB]_{ii} = \sum_{i,j} a_{ij}b_{ji}$
$\text{trace}(BA) = \sum_i [BA]_{ii} = \sum_{i,j} b_{ij}a_{ji}$))

so $\text{trace}(ASA^{-1}) = \text{trace}(SA^{-1}A) = \text{trace}\, S$.

$K :=$ Gauss curvature $= \det(S)$ $S =$ Shape operator } has an interesting interpretation in terms of area ratios for the normal map $U: M^2 \to S^2$, called the Gauss map.

$H :=$ Mean curvature $= \frac{1}{2}\,\text{trace}(S)$ } measures average concavity of surface at P ~ see Hessian discussion above

For the shape operator,
$K = k_1 k_2$, $H = \frac{1}{2}(k_1 + k_2)$ <u>And</u> k_1, k_2 are roots of $|\lambda I - A| = 0$
$\lambda^2 - 2H\lambda + K = 0$.