Math 4530
Fri 11 Feb

Hw for Fri 2/18
Chapter 2:
(2.15)
(3.4, 3.9, 3.10, 3.11) 3.12
(4.4, 4.6, 4.8)

(circled means hand in). others just recommended.

- page 14 wed
  - bases, coords,
  - linear transformations
  - matrices of linear transformations.

page 14 did case of $\dim V = 2$.
In general:

Matrix of a linear transformation

$L: V \to V$ linear, $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ a basis for V

if $\vec{v} \in V$, $\vec{v} = \sum_k^1 a^k \vec{v}_k$ then the coordinates of $\vec{v}$ wrt $\mathcal{B}$ are

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} a^1 \\ a^2 \\ \vdots \\ a^n \end{bmatrix}$$

Then $L(\vec{v}) = L\left(\sum a^k \vec{v}_k\right)$

$$= \sum a^k L(\vec{v}_k)$$

So

$$[L(\vec{v})]_{\mathcal{B}} = \sum^1 a^k [L(\vec{v}_k)]_{\mathcal{B}}$$

$$= \underbrace{\begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{B}} & [L(\vec{v}_2)]_{\mathcal{B}} & \cdots & [L(\vec{v}_n)]_{\mathcal{B}} \end{bmatrix}}_{:= \text{ the matrix of } L \text{ wrt } \mathcal{B},\ [L]_{\mathcal{B}}} \begin{bmatrix} a^1 \\ a^2 \\ \vdots \\ a^n \end{bmatrix}$$

The shape operator is not only linear, it is also <u>self-adjoint</u>

<u>Def</u>: If $\vec{V}$ is a vector space with an inner product (dot product).
Then $L: V \to V$ linear is self adjoint iff

$$L(\vec{v}) \cdot \vec{w} = \vec{v} \cdot L(\vec{w}) \qquad \forall\, \vec{v}, \vec{w} \in X$$

<u>Theorem</u>: let $L: V \to V$ self adjoint, $\dim V = n < \infty$. This condition is equivalent to any of the following 3 statements:

(1) $\exists$ a basis $\{\vec{v}_1, \ldots \vec{v}_n\}$ for V s.t. $L(\vec{v}_i) \cdot \vec{v}_j = \vec{v}_i \cdot L(\vec{v}_j) \quad \forall\, i, j$.

(2) $\exists$ an ortho-normal basis $\underbrace{\{\vec{f}_1, \ldots \vec{f}_n\}}_{= \mathcal{B}}$ for V s.t. the matrix $[L]_{\mathcal{B}}$ is symmetric,
i.e. $b_{ij} = b_{ji}$
$B = B^T$

(3) For every o.n. basis $\mathcal{B} = \{\vec{f}_1, \ldots \vec{f}_n\}$ of V, the matrix $[L]_{\mathcal{B}}$ is symmetric.

<u>proof</u>: L self adjoint $\Rightarrow$ (1), special case
But (1) $\Rightarrow L$ self adjoint: Let $\vec{v} = \sum a^k \vec{v}_k$
$\vec{w} = \sum b^j \vec{v}_j$

$$L(\vec{v}) \cdot \vec{w} = \sum_k a^k L(\vec{v}_k) \cdot \sum_j b^j \vec{v}_j = \sum_{k,j} a^k b^j L(\vec{v}_k) \cdot \vec{v}_j$$

$$\vec{v} \cdot L(\vec{w}) = \left(\sum_k a^k \vec{v}_k\right) \cdot \left(\sum_j b^j L(\vec{v}_j)\right) = \sum_{k,j} a^k b^j \vec{v}_k \cdot L(\vec{v}_j)$$

∎

Now, for an o.n. basis $\{\vec{f}_1, \ldots \vec{f}_n\}$

$$L(\vec{f}_j) = \sum_i (L(\vec{f}_j) \cdot \vec{f}_i)\, \vec{f}_i$$

so $\text{entry}_{ij}([L]_{\mathcal{B}}) = L(\vec{f}_j) \cdot \vec{f}_i$

Thus self adjoint $\Rightarrow$ (3).
But (3) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ self adjoint ∎
(↑ by 1st discussion)

Theorem : The Shape operator is self adjoint.

proof: Let $X: D \to \mathcal{U} \subset M$ a patch

so $\{\vec{X}_u, \vec{X}_v\}$ a basis for T_pM.

By (1) of Last theorem, we need only check $S(\vec{X}_u)\cdot\vec{X}_v = \vec{X}_u \cdot S(\vec{X}_v)$.
(others automatic).

$$S(X_u) = -\nabla_{\vec{X}_u}\vec{U} = -U_u$$

so we are checking $-\vec{U}_u \cdot \vec{X}_v \overset{?}{=} \vec{X}_u \cdot (-\vec{U}_v)$

i.e. $\vec{U}_u \cdot \vec{X}_v = \vec{U}_v \cdot \vec{X}_u$?

but $U \cdot X_u \equiv 0$, $U \cdot X_v \equiv 0$ on patch,

so $U_v \cdot X_u + U \cdot X_{uv} \equiv 0$, $U_u \cdot X_v + U \cdot X_{vu} \equiv 0$ $\Rightarrow U_v \cdot X_u = U_u \cdot X_v$!

Theorem (Spectral Theorem)

If $L: V \to V$ is self adjoint then

there is an orthonormal basis for V made out of eigenvectors of L (an eigenbasis)

Def: For the shape operator the eigenvector directions are called principal directions; eigenvalues are called principal curvatures

What are the principal directions on the sphere, plane, cylinder?

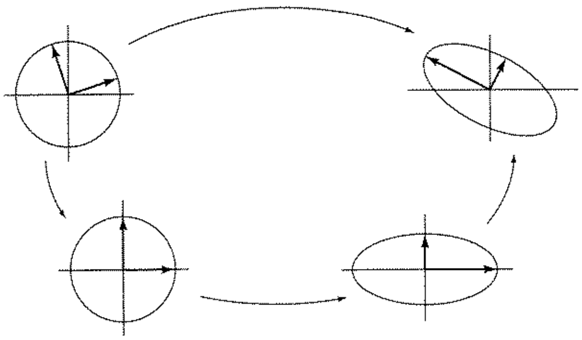
$\frac{1}{2}(\lambda_1+\lambda_2)$ = Mean curvature

$\lambda_1\lambda_2$ = Gauss curvature.

proof of spectral theorem: You prove the $n=2$ case in HW.

In general this is a topic in 2270, which we could discuss if you wish.

CHAPTER 8



Symmetric Matrices and Quadratic Forms

8.1 SYMMETRIC MATRICES

> In this chapter we will work with real numbers throughout, except for a brief digression into $\mathbb{C}$ in the discussion of Fact 8.1.3.

Our work in the last chapter dealt with the following central question:

> When is a given square matrix A *diagonalizable?* That is, when is there an *eigenbasis* for A?

In geometry, we prefer to work with *orthonormal* bases, which raises the following question:

> For which matrices is there an *orthonormal* eigenbasis? Or, equivalently, for which matrices A is there an *orthogonal* matrix S such that $S^{-1}AS = S^TAS$ is diagonal?

(Recall that $S^{-1} = S^T$ for orthogonal matrices, by Fact 5.3.7.) We say that A is *orthogonally diagonalizable* if there exists an orthogonal S such that $S^{-1}AS = S^TAS$ is diagonal. Then, the question is

> Which matrices are orthogonally diagonalizable?

Simple examples of orthogonally diagonalizable matrices are diagonal matrices (we can let $S = I_n$) and the matrices of orthogonal projections and reflections.

EXAMPLE 1 If A is orthogonally diagonalizable, what is the relationship between A^T and A?

Solution

We have

$$S^{-1}AS = D \quad \text{or} \quad A = SDS^{-1} = SDS^T,$$

for an orthogonal S and a diagonal D. Then

$$A^T = (SDS^T)^T = SD^T S^T = SDS^T = A.$$

We find that A is symmetric:

$$A^T = A.$$

Surprisingly, the converse is true as well:

Fact 8.1.1

Spectral theorem

A matrix A is *orthogonally diagonalizable* (i.e., there exists an orthogonal S such that $S^{-1}AS = S^T AS$ is diagonal) if and only if A is *symmetric* (i.e., $A^T = A$).

We will prove this theorem later in this section, based on two preliminary resul Facts 8.1.2 and 8.1.3. First, we will illustrate the spectral theorem with an examp

EXAMPLE 2 For the symmetric matrix $A = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$, find an orthogonal S such that $S^{-1}AS$ diagonal.

Solution

We will first find an eigenbasis. The eigenvalues of A are 3 and 8, with correspon ing eigenvectors $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, respectively. (See Figure 1.)

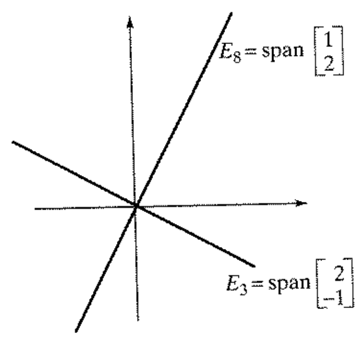


Figure 1

Note that the two eigenspaces, E_3 and E_8, are perpendicular. (This is no incidence, as we will see in Fact 8.1.2.) Therefore, we can find an orthonor eigenbasis simply by dividing the given eigenvectors by their lengths:

$$\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

If we define the orthogonal matrix

$$S = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix},$$

then $S^{-1}AS$ will be diagonal, namely, $S^{-1}AS = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$.

The key observation we made in Example 2 generalizes as follows:

Fact 8.1.2

> Consider a symmetric matrix A. If $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of A with *distinct* eigenvalues λ_1 and λ_2, then $\vec{v}_1 \cdot \vec{v}_2 = 0$; that is, $\vec{v}_2$ is orthogonal to $\vec{v}_1$.

Proof We compute the product

$$\vec{v}_1^T A \vec{v}_2$$

in two different ways:

$$\vec{v}_1^T A \vec{v}_2 = \vec{v}_1^T(\lambda_2 \vec{v}_2) = \lambda_2(\vec{v}_1 \cdot \vec{v}_2)$$
$$\vec{v}_1^T A \vec{v}_2 = \vec{v}_1^T A^T \vec{v}_2 = (A\vec{v}_1)^T \vec{v}_2 = (\lambda_1 \vec{v}_1)^T \vec{v}_2 = \lambda_1(\vec{v}_1 \cdot \vec{v}_2)$$

Comparing the results, we find

$$\lambda_1(\vec{v}_1 \cdot \vec{v}_2) = \lambda_2(\vec{v}_1 \cdot \vec{v}_2),$$

or

$$(\lambda_1 - \lambda_2)(\vec{v}_1 \cdot \vec{v}_2) = 0.$$

Since the first factor in this product, $\lambda_1 - \lambda_2$, is nonzero, the second factor, $\vec{v}_1 \cdot \vec{v}_2$, must be zero, as claimed. ∎

Fact 8.1.2 tells us that the eigenspaces of a symmetric matrix are perpendicular to one another. Here is another illustration of this property:

EXAMPLE 3 For the symmetric matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

find an orthogonal S such that $S^{-1}AS$ is diagonal.

Solution

The eigenvalues are 0 and 3, with

$$E_0 = \operatorname{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right) \quad \text{and} \quad E_3 = \operatorname{span}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Note that the two eigenspaces are indeed perpendicular to one another, in accordance with Fact 8.1.2. (See Figure 2.)

We can construct an orthonormal eigenbasis for A by picking an orthonormal basis of each eigenspace (using the Gram–Schmidt process in the case of E_0). See Figure 3.

In Figure 3, the vectors $\vec{v}_1$, $\vec{v}_2$ form an orthonormal basis of E_0, and $\vec{v}_3$ is a unit vector in E_3. Then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is an orthonormal eigenbasis for A. We can let $S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$ to diagonalize A orthogonally.

If we apply the Gram–Schmidt[1] process to the vectors

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

[1] Alternatively, we could find a unit vector $\vec{v}_1$ in E_0 and a unit vector $\vec{v}_3$ in E_3, and then let $\vec{v}_2 = \vec{v}_3 \times$

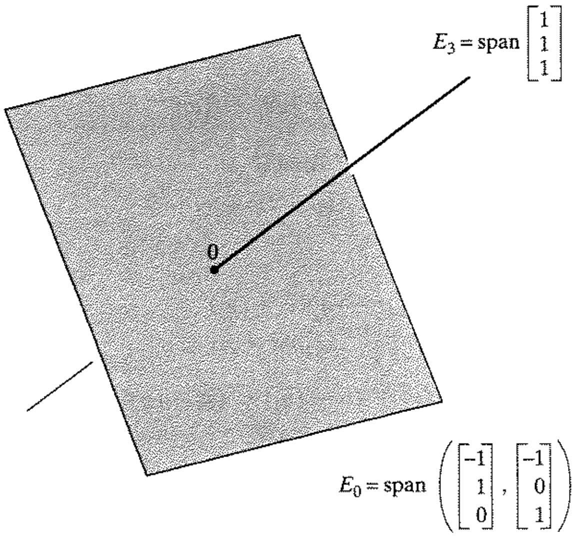


Figure 2 The eigenspaces E_0 and E_3 are orthogonal complements.

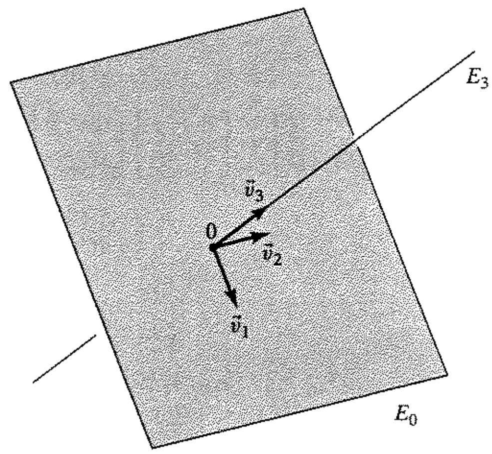


Figure 3

spanning E_0, we find

$$\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\-1\\2\end{bmatrix}.$$

The computations are left as an exercise. For E_3, we get

$$\vec{v}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

Therefore, the orthogonal matrix

$$S = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

diagonalizes the matrix A:

$$S^{-1}AS = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

■

By Fact 8.1.2, if a symmetric matrix is diagonalizable, then it is orthogonally diagonalizable. We still have to show that symmetric matrices are diagonalizable in the first place (over $\mathbb{R}$). The key point is the following observation:

Fact 8.1.3 A symmetric $n \times n$ matrix A has n real eigenvalues if they are counted with their algebraic multiplicities.

Proof (This proof is for those who have studied Section 7.5.) By Fact 7.5.4, we need to show that all the complex eigenvalues of matrix A are in fact real. Consider two complex conjugate eigenvalues $p \pm iq$ of A with corresponding eigenvectors $\vec{v} \pm i\vec{w}$.

(Compare this with Exercise 7.5.42b.) We wish to show that these eigenvalues are real; that is, $q = 0$. Note first that

$$(\vec{v} + i\vec{w})^T(\vec{v} - i\vec{w}) = \|\vec{v}\|^2 + \|\vec{w}\|^2.$$

(Verify this.) Now we compute the product

$$(\vec{v} + i\vec{w})^T A(\vec{v} - i\vec{w})$$

in two different ways:

$$\begin{aligned}(\vec{v} + i\vec{w})^T A(\vec{v} - i\vec{w}) &= (\vec{v} + i\vec{w})^T(p - iq)(\vec{v} - i\vec{w}) \\ &= (p - iq)(\|\vec{v}\|^2 + \|\vec{w}\|^2) \\ (\vec{v} + i\vec{w})^T A(\vec{v} - i\vec{w}) &= \left(A(\vec{v} + i\vec{w})\right)^T(\vec{v} - i\vec{w}) = (p + iq)(\vec{v} + i\vec{w})^T(\vec{v} - i\vec{w}) \\ &= (p + iq)(\|\vec{v}\|^2 + \|\vec{w}\|^2).\end{aligned}$$

Comparing the results, we find that $p + iq = p - iq$, so that $q = 0$, as claimed. ■

The foregoing proof is not very enlightening. A more transparent proof would follow if we were to define the dot product for complex vectors, but to do so would lead us too far afield.

We are now ready to prove Fact 8.1.1: Symmetric matrices are orthogonally diagonalizable.

Even though this is not logically necessary, let us first examine the case of a symmetric $n \times n$ matrix A with n distinct real eigenvalues. For each eigenvalue, we can choose an eigenvector of length 1. By Fact 8.1.2, these eigenvectors will form an orthonormal eigenbasis, that is, the matrix A will be orthogonally diagonalizable, as claimed.

Proof (of Fact 8.1.1): This proof is somewhat technical; it may be skipped in a first reading of this text without harm.

We prove by induction on n that a symmetric $n \times n$ matrix A is orthogonally diagonalizable. (See Fact 6.1.7.)

For a 1×1 matrix A, we can let $S = [1]$.

Now assume that the claim is true for $n - 1$; we show that it holds for n. Pick a real eigenvalue λ of A (this is possible by Fact 8.1.3), and choose an eigenvector $\vec{v}_1$ of length 1 for λ. We can find an orthonormal basis $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ of $\mathbb{R}^n$. (Think about how you could construct such a basis.) Form the orthogonal matrix

$$P = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix},$$

and compute

$$P^{-1}AP.$$

The first column of $P^{-1}AP$ is $\lambda\vec{e}_1$. (Why?) Also note that $P^{-1}AP = P^TAP$ is symmetric: $(P^TAP)^T = P^TA^TP = P^TAP$, because A is symmetric. Combining these two statements, we conclude that $P^{-1}AP$ is of the form

$$P^{-1}AP = \begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix}, \tag{I}$$

where B is a symmetric $(n-1)\times(n-1)$ matrix. By induction hypothesis, we assume that B is orthogonally diagonalizable; that is, there is an orthogonal $(n-1)\times(n-1)$ matrix Q such that

$$Q^{-1}BQ = D$$

is a diagonal $(n-1) \times (n-1)$ matrix. Now introduce the orthogonal $n \times n$ matrix

$$R = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}.$$

Then

$$R^{-1}\begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix}R = \begin{bmatrix} 1 & 0 \\ 0 & Q^{-1} \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & D \end{bmatrix} \qquad \textbf{(II)}$$

is diagonal.

Combining equations (I) and (II), we find that

$$R^{-1}P^{-1}APR = \begin{bmatrix} \lambda & 0 \\ 0 & D \end{bmatrix} \qquad \textbf{(III)}$$

is diagonal. Consider the orthogonal matrix $S = PR$. (Recall Fact 5.3.4a: The product of orthogonal matrices is orthogonal.) Note that $S^{-1} = (PR)^{-1} = R^{-1}P^{-1}$. Therefore, equation (III) can be written

$$S^{-1}AS = \begin{bmatrix} \lambda & 0 \\ 0 & D \end{bmatrix},$$

proving our claim. ■

The method outlined in the proof of Fact 8.1.1 is not a sensible way to find the matrix S in a numerical example. Rather, we can proceed as in Example 3:

Fact 8.1.4

Orthogonal diagonalization of a symmetric matrix A

a. Find the eigenvalues of A, and find a basis of each eigenspace.
b. Using the Gram–Schmidt process, find an *orthonormal* basis of each eigenspace.
c. Form an orthonormal eigenbasis $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ for A by concatenating the orthonormal bases you found in part (b), and let

$$S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix}.$$

S is orthogonal (by Fact 8.1.2), and $S^{-1}AS$ will be diagonal.

We conclude this section with an example of a geometric nature:

EXAMPLE 4

Consider an invertible symmetric 2×2 matrix A. Show that the linear transformation $T(\vec{x}) = A\vec{x}$ maps the unit circle into an ellipse, and find the lengths of the semimajor and the semiminor axes of this ellipse in terms of the eigenvalues of A. Compare this with Exercise 2.2.50.

Solution

The spectral theorem tells us that there is an orthonormal eigenbasis $\vec{v}_1, \vec{v}_2$ for T, with associated real eigenvalues λ_1 and λ_2. Suppose that $|\lambda_1| \geq |\lambda_2|$. These eigenvalues will be nonzero, since A is invertible. The unit circle in $\mathbb{R}^2$ consists of all vectors of the form

$$\vec{v} = \cos(t)\vec{v}_1 + \sin(t)\vec{v}_2.$$

The image of the unit circle consists of the vectors

$$T(\vec{v}) = \cos(t)T(\vec{v}_1) + \sin(t)T(\vec{v}_2)$$
$$= \cos(t)\lambda_1\vec{v}_1 + \sin(t)\lambda_2\vec{v}_2,$$

an ellipse whose semimajor axis $\lambda_1\vec{v}_1$ has the length $\|\lambda_1\vec{v}_1\| = |\lambda_1|$, while the length of the semiminor axis is $\|\lambda_2\vec{v}_2\| = |\lambda_2|$. (See Figure 4.)

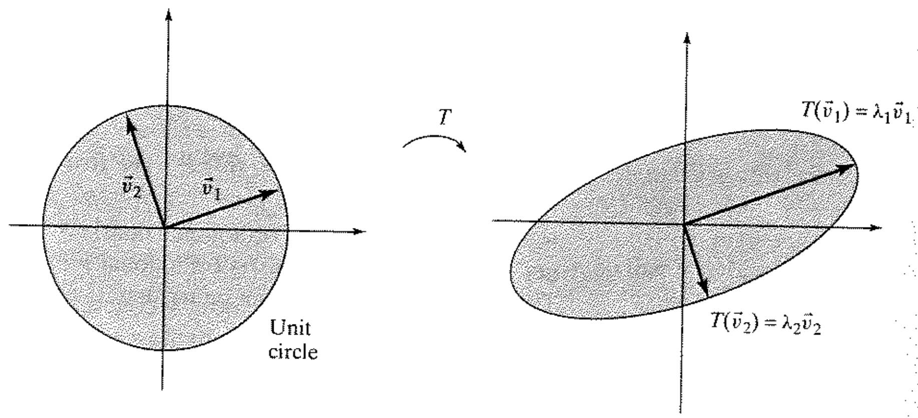


Figure 4

In the example illustrated in Figure 4, the eigenvalue λ_1 is positive, and λ_2 is negative. ■

EXERCISES 8.1

GOAL *Find orthonormal eigenbases for symmetric matrices. Apply the spectral theorem.*

For each of the matrices in Exercises 1 through 6, find an orthonormal eigenbasis. Do not use technology.

1. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$

For each of the matrices A in Exercises 7 through 11, find an orthogonal matrix S and a diagonal matrix D such that $S^{-1}AS = D$. Do not use technology.

7. $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

8. $A = \begin{bmatrix} 3 & 3 \\ 3 & -5 \end{bmatrix}$

9. $A = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

10. $A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix}$

11. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

12. Let L from $\mathbb{R}^3$ to $\mathbb{R}^3$ be the reflection about the line spanned by

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

a. Find an orthonormal eigenbasis $\mathfrak{B}$ for L.

b. Find the matrix B of L with respect to $\mathfrak{B}$.

c. Find the matrix A of L with respect to the standard basis of $\mathbb{R}^3$.

13. Consider a symmetric 3×3 matrix A with $A^2 = I_3$. Is the linear transformation $T(\vec{x}) = A\vec{x}$ necessarily the reflection about a subspace of $\mathbb{R}^3$?

14. In Example 3 of this section, we diagonalized the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

by means of an orthogonal matrix S. Use this result to diagonalize the following matrices orthogonally (find S and D in each case):