Math 4530
Wed 4/6

Minimal surfaces & complex analysis

We'll be following some old undergraduate colloquium notes, filling in details with supplementary notes numbered accordingly...

Conformal Maps

4 extra

Def: A patch $X: D \to M^2$ ($D \subset \mathbb{R}^2$) is called isothermal or conformal iff the matrix of the 1^{st} fdl form is diagonal, i.e.

$$[g_{ij}] = \varphi^2 \cdot [Id] \quad ; \quad |X_u| = |X_v| = \varphi, \quad X_u \cdot X_v = 0$$

example: inverse stereographic projection

Def: $F: M^2 \to N^2$ is a conformal map iff $F_*: T_pM \to T_{f(p)}N$ is angle preserving

i.e. $\measuredangle \vec{v}, \vec{w} = \measuredangle F_*\vec{v}, F_*\vec{w}$
$\forall \vec{v}, \vec{w} \in T_pM$
$\forall p \in M$

HW1 (for 4/15) Show F is conformal at p iff the differential map F_* takes squares to squares

Precisely: Let $\{\vec{f}_1, \vec{f}_2\}$ an orthonormal basis for T_pM

Show F is conformal at p iff $\begin{cases} F_*\vec{f}_1 \cdot F_*\vec{f}_2 = 0 \\ |F_*\vec{f}_1| = |F_*\vec{f}_2| \end{cases}$

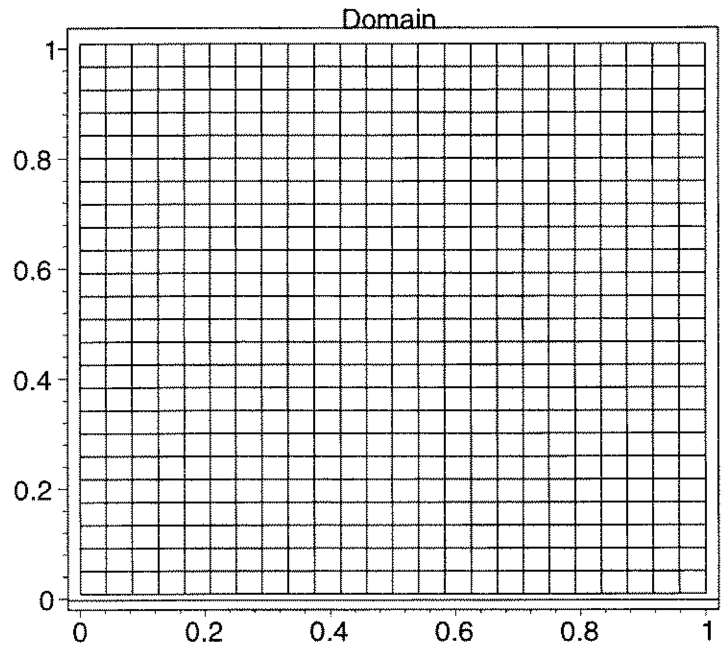
HW2 Show F is conformal at p iff

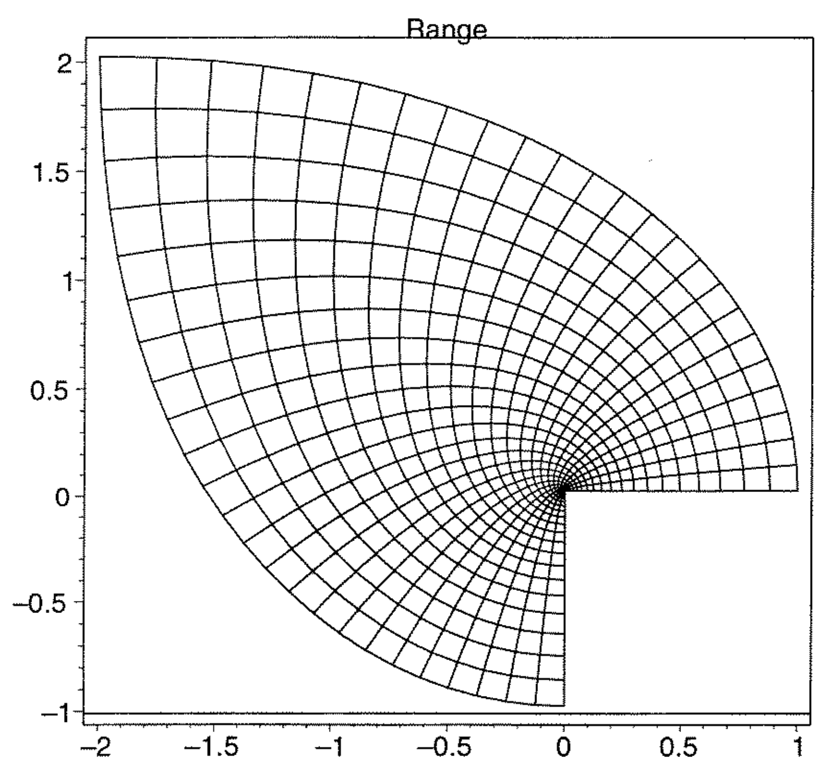
$$F_*\vec{v} \cdot F_*\vec{w} = \varphi^2(p)\, \vec{v} \cdot \vec{w}$$ for some scalar $\varphi^2(p) > 0$, $\forall \vec{v}, \vec{w} \in T_pM$.

Thus, conformal patches are special cases of conformal maps

6 extra

HW3) For the complex analytic function g(z)=z^3, and the point P=1+i, write the affine approximation for g at P in complex form and in real form, as is done for g(z)=z^2 on page 6 of the Undergraduate Colloquium lecture notes on minimal surfaces. Also draw the "infinitesimal square" picture as is done there. The picture should reflect the scaling and rotation which is done to tangent vectors at P, in order to get their images at g(P). You may use these pictures:





7 extra

<u>Theorem</u> If X is a conformal patch for any surface,
if $U = \frac{X_u \times X_v}{|X_u \times X_v|}$

Then $\boxed{X_{uu} + X_{vv} = 2H\varphi^2 \vec{U}}$ (H = mean curvature)

In particular,
$\boxed{X_{uu} + X_{vv} = \vec{0}}$ iff M^2 is minimal.

pf: Let $X_u \cdot X_u = X_v \cdot X_v = \varphi^2(u,v)$
$X_u \cdot X_v \equiv 0.$

$X_{uu} \cdot X_u = \frac{1}{2}\varphi^2_u$

$X_{vv} \cdot X_u = \underbrace{(\cancel{X_v \cdot X_u})_v}_{0} - X_v \cdot X_{uv} = -\frac{1}{2}(X_v \cdot X_v)_u = -\frac{1}{2}\varphi^2_u$

thus
$(X_{uu} + X_{vv}) \cdot X_u = 0$
symmetry $\Rightarrow (X_{uu} + X_{vv}) \cdot X_v = 0$
so $X_{uu} + X_{vv} \parallel \vec{U}.$

$X_{uu} \bullet \vec{U} = \underbrace{(\cancel{X_u \cdot \vec{U}})_u}_{0} \underbrace{- X_u \cdot U_u}_{\underbrace{-X_u \cdot (-a_1^1 X_u - a_1^2 X_v)}_{a_1^1 \varphi^2}}$

so also $X_{vv} \cdot U = a_2^2 \varphi^2$

$\Rightarrow X_{uu} + X_{vv} = 2H\varphi^2 \vec{U}$

∎

<u>Theorem</u> (See page 8): Let M^2 be a minimal surface.
Then the Gauss map $U: M^2 \to S^2$ is conformal

pf: Let $\{\vec{f}_1, \vec{f}_2\}$ be o.n. principle directions at $p \in M^2$, minimal.

$S(\vec{f}_1) = k_1 \vec{f}_1$
$S(\vec{f}_2) = -k_1 \vec{f}_1$ since $H=0$

but recall (Feb16), for $\vec{v} = \alpha'(0) \in T_pM$

$$S(\vec{v}) = -\nabla_{\vec{v}} \vec{U} = -\frac{d}{dt} \vec{U} \circ \alpha(t)\Big|_{t=0} = -U_*(\vec{v})$$

Thus $U_*(\vec{f}_1) = -k_1 \vec{f}_1$ $\quad U_* \vec{f}_1 \cdot U_* \vec{f}_2 = 0$
$U_*(\vec{f}_2) = k_1 \vec{f}_2$ $\quad |U_* \vec{f}_1| = |U_* \vec{f}_2| = |k_1|$

So by HW1, U is conformal, ~~fact~~ with stretching factor $\varphi = |k_1| = \sqrt{-K}$.

<u>Def</u>: Let M^2, N^2 be oriented (global $\vec{U}$'s).
$F: M \to N$
F is <u>orientation preserving</u> at $p \in M$ iff
positively oriented triples $\{\underbrace{\vec{f}_1, \vec{f}_2}_{\text{basis for } T_pM}, \vec{U}_p\}$ are mapped to positively oriented triples $\{F_*\vec{f}_1, F_*\vec{f}_2, U_{F(p)}\}$

In the reverse case, F is <u>orientation reversing</u>

8 extra extra

<u>Theorem</u> : If M is minimal then $\vec{U}: M^2 \to S^2$ is <u>orientation reversing</u>
(with respect to the outer normal $\vec{U}$ on S^2)

proof : If $\{\underbrace{\vec{f}_1, \vec{f}_2}_{\text{principle dirs}}, \vec{U}\}$ is positively oriented at p (det>0), then

$\{-k_1\vec{f}_1, k_1\vec{f}_2, \vec{U}\}$ is negatively oriented! ∎

<u>Theorem</u> St^{-1} (Hence St) is <u>orientation reversing</u> if $\vec{U}$ = outer normal on S^2
$\vec{U} = \vec{e}_3$ on $\mathbb{R}^2$.

<u>pf</u> $St^{-1}(u,v) = X(u,v) = \langle \frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{u^2+v^2+1} \rangle$

at $(u,v)=(0,0)$

$X(0,0) = \langle 0,0,-1 \rangle$ = South pole

$$\left.\begin{aligned} X_*(\vec{e}_1) &= X_u = \langle 2,0,0 \rangle \\ X_*(\vec{e}_2) &= X_v = \langle 0,2,0 \rangle \\ \vec{U}(0,0,-1) &= \langle 0,0,-1 \rangle \end{aligned}\right\} \{X_*(\vec{e}_1), X_*(\vec{e}_2), \vec{U}\} \text{ negatively oriented !}$$

but $\det\{X_u|X_v|U\} \neq 0$, cont on $\mathbb{R}^2$, negative when $(u,v)=(0,0)$
$\Rightarrow < 0$ everywhere ∎

<u>Corollary</u> : $g := St \circ U \circ X$ is conformal and <u>orientation preserving</u>

if $U = \frac{X_u \times X_v}{|X_u \times X_v|}$ on M

U = outer normal on S^2

$U = \vec{e}_3$ on $\mathbb{R}^2$

<u>And</u> has a computable expansion factor φ, as the product of the expansion factors of its 3 intermediate conformal maps

<u>proof</u> : $(G \circ F)_* \vec{v} = G_*(F_* \vec{v})$ (chain rule! if $\alpha(0)=p$, $\alpha'(0)=v$

$F_* v = (F \circ \alpha)'(0)$

$G_*(F_* v) = (G \circ (F \circ \alpha))'(0)$

$= ((G \circ F) \circ \alpha)'(0)$

$= (G \circ F)_* \vec{v}$)

Thus compositions of orientation preserving maps are orientation preserving, and so are compositions of two orientation reversing ones!

Also the scaling factor for the composition of two conformal maps is the product of the scaling factors!