Math 4630
Wed 27 April

Global Gauss-Bonnet Theorem

Let \( M^2 \) be an oriented surface, possibly with boundary curve(s), \( \partial M \).

Then

\[
\oint_{\partial M} k g \, ds + \iint_M K dA = 2\pi \chi(M)
\]

where \( \chi(M) \) is the Euler characteristic of \( M \):

For a triangulation of \( M \), let

- \( F \) = number of triangle faces
- \( E \) = number of edges
- \( V \) = number of vertices

\[
\chi(M) := V - E + F
\]

1. Each triangle has 3 distinct edges.
2. Each edge is an edge of exactly 2 triangles, unless it is a boundary edge (of a single triangle).
3. Each edge has 2 distinct vertices.

(In fact, one can consider more general "polygons", as long as it can be decomposed into a triangulation without changing \( \chi \)).

Examples:

- \( S^2 \)
- \( T^2 \) (torus)
- Piece of cylinder
- "Flat torus"

Check:

\[
\begin{align*}
\chi(S^2) &= 2 \\
\chi(T^2) &= 0 \\
\chi(T^1 \times T^1) &= -2 \\
\text{double torus,} & \text{ "connect sum" of two tori.}
\end{align*}
\]
Consequences of G.B. (for \( \partial M = \emptyset \))

1. \( \chi(M) \) is independent of triangulation
2. \( \iint_{M} K dA \) is independent of shape (diffeomorphism) of given surface.

Proof of global G.B.

For each triangle \( T_i \):

\[
\oint_{T_i} k g d\sigma + \sum_{\text{corners}} (\pi - \theta_i) + \iint_{T_i} K dA = 2\pi
\]

i.e.

\[
\oint_{T_i} k g d\sigma + \iint_{T_i} K dA = -\pi + \sum \theta_i
\]

Add these identities, sum over all triangles.

Interior edges \( \oint_{T_i} k g d\sigma \) cancel, because \( \tau \) changes sign if you traverse curve in opposite direction.

When you sum over all triangles, the interior vertex angles add up to \( 2\pi \) at each vertex, along boundary vertices you get \( \pi \).

Yields:

\[
\oint_{\partial M} k g d\sigma + \iint_{M} K dA = -\pi F + 2\pi V_{\text{int}} + \pi V_{\partial}
\]

\( V_{\text{int}} \) \# interior vertices \( V_{\partial} \) \# boundary vertices

By triangulation hypotheses,

\[
3F = 2E_{\text{int}} + E_{\partial} = 2E - E_{\partial}
\]

i.e. \( E = \frac{3}{2} F + \frac{1}{2} E_{\partial} \)

So:

\[
-\pi F + 2\pi V_{\text{int}} + \pi V_{\partial} = 2\pi \left( E_{\partial} + \frac{1}{2} V_{\partial} \right)
\]

\[
= 2\pi \left( F - E + \frac{1}{2} E_{\partial} - \frac{1}{2} V_{\partial} \right)
\]

\[
= 2\pi \chi(M)
\]
The Greeks knew that there are exactly 5 regular polyhedra (with flat polygonal faces).

For us, a regular polyhedron will be a polygonalization of a surface in which each face is an $n$-gon (same $n$, all faces) and s.t. at each vertex exactly $k$ polygons meet. (Same $k$, all vertices)

Call such a polyhedron an $(n,k)$ polyhedron. (Our polygons need not be flat; the Greeks' were.)

**Examples**

![Examples](image)

$(3,3)$  

$(4,3)$

12a) Prove that the sphere has only 5 regular polygonizations (the Greeks did).

Use Euler characteristic $= 2$.

**Hint**: recall, in 6.8, for $\Delta$'s we had the identity $3F = 2E$.

For an $(n,k)$ polygonation you have a similar identity,

as well as one relating faces $F$ to vertices $V$.

Hence,$$
\chi = 2\pi (F - E + V) = 2\pi F (1 - \frac{1}{n} + \frac{1}{k}) = 2
$$

↑

only 5 choices of $(n,k)$ work; you can list them by exhaustion, and check they work.

12b) Show that the torus has only two types of regular $(n,k)$ polygonizations, (although $F$ is indeterminate). Draw a picture of each type.

[They can't be realized in $\mathbb{R}^3$ with flat polygons.]
Gauss-Bonnet for paper surfaces with straight-line edges:

Recall, we defined \[ \sum K dA = 2\pi - \Theta \text{ at the vertex} \]

\[ \theta_1 \quad \theta_2 \quad \theta_3 \]

2\pi - (\theta_1 + \theta_2 + \theta_3).

Let M be flat, except possibly at vertices.

Assume M has no boundary. (Could extend to this case.)

Then

\[ \sum (2\pi - \Theta) = 2\pi \chi(M) \]

pf: (for a triangulation):

\[ \text{LHS } = 2\pi V - \sum \Theta \text{ at vertex} \]

\[ = 2\pi V - \sum \pi \text{ triangle} \]

\[ = 2\pi V - \pi F \]

\[ = 2\pi (V - E + F) \quad \text{since } 3F = 2E \]

Example:

\[ \chi(S^2) \]

\[ \chi(\mathbb{R}^3) = 4\pi - 2\pi \cdot 2 \]

Example: Flat torus:

2 vertices, no angle excess

\[ \Rightarrow \chi = 0. \]

Example: \( T^2 \)

16 vertices

For 8, \( \sum K dA = \pi \frac{2}{2} ! \)

For 8, \( \sum K dA = -\pi \frac{2}{2} ! \)