Math 4530
Wed 27 April

Final HW is due
on Wed 10:30 ~ start of final exam.
optional for project people
40% of final exam for the rest!

Global Gauss-Bonnet Theorem

Let M^2 be an oriented surface, possibly with boundary curve(s), ∂M.
Then

$$\boxed{\oint_{\partial M} k_g\, ds + \iint_M K\, dA = 2\pi \chi(M)}$$

where $\chi(M)$ is the Euler characteristic of M:

For a triangulation of M, let F = # triangle faces
E = # edges
V = # vertices

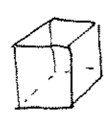
$$\chi(M) := V - E + F$$

1. each Δ has 3 distinct edges
2. each edge is an edge of exactly 2 triangles, unless it is a boundary edge (of a single triangle)
3. each edge has 2 distinct vertices.

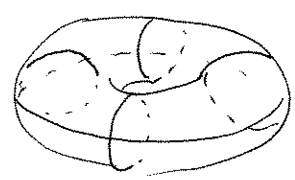
(In fact, one can consider more general "polygonations", as long as it can be decomposed into a triangulation without changing χ).

examples

S^2

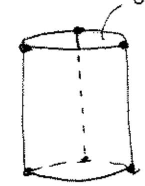


S^2



T^2

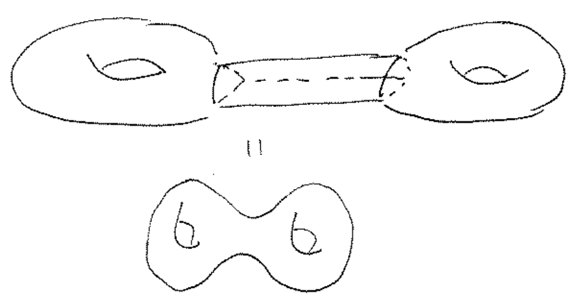
open top & bottom



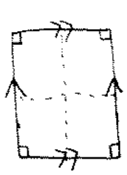
piece of cylinder.

check: $\chi(S^2) = 2$ $\qquad \chi(cyl) = 0$
$\chi(T^2) = 0$
$\chi(T^2 \# T^2) = -2$

$\downarrow$
double torus,
"connect sum"
of two tori.



"flat torus"



$ds = ds_{\mathbb{E}}$

Consequences of G.B. (for $\partial M = \emptyset$)

(1) $\chi(M)$ is independent of triangulation

(2) $\iint_M K\,dA$ is independent of shape (diffeomorphism) of given surface.

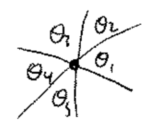
proof of global G.B.

for each triangle T_i

$$\oint_{\partial T_i} k_g\,ds + \sum_{\text{corners}} (\pi - \theta_i) + \iint_{T_i} K\,dA = 2\pi$$

i.e.

$$\oint_{\partial T_i} k_g\,ds + \iint_{T_i} K\,dA = -\pi + \sum \theta_i$$



add these identities, sum over all triangles.

Interior edges $\oint_{\partial T_i} k_g\,ds$ cancel, because $\vec{n}$ changes sign if you traverse curve in opposite direction.

when you sum over all triangles, the interior vertex angles add up to 2π at each vertex, along boundary vertices you get π.

yields: $$\oint_{\partial M} k_g\,ds + \iint_M K\,dA = -\pi F + 2\pi \underbrace{V_{int}}_{\#\text{ interior vertices}} + \pi \underbrace{V_\partial}_{\#\text{ bdry vertices}}$$

by triangulation hypotheses,

$$3F = 2E_{int} + E_\partial = 2E - E_\delta$$

i.e. $E = \frac{3}{2}F + \frac{1}{2}E_\delta$

so $-\pi F + 2\pi V_{int} + \pi V_\partial$

$$= 2\pi\left(-\frac{F}{2} + V - \frac{1}{2}V_\partial\right)$$

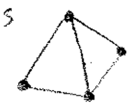
$$= 2\pi\left(F - E + V + \underbrace{\tfrac{1}{2}E_\delta - \tfrac{1}{2}V_\delta}_{=0!}\right)$$

$$= 2\pi\chi(M)$$

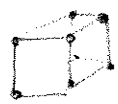
The Greeks know that there are exactly 5 regular polyhedra (with flat polygonal faces).

For us, a regular polyhedron will be a polygonation of a surface
in which each face is an n-gon (same n, all faces)
and s.t. at each vertex exactly k polygons meet. (same k, all vertices)
Call such a polyhedron an (n,k) polyhedron. (Our polygons need not be flat; the Greeks' were.)

examples



(3,3)



(4,3)

... ?

12a Prove that the sphere has only 5 regular polygonations (the Greeks')
Use Euler characteristic = 2

Hint : recall, in G.B., for Δ's we had the identity $3F = 2E$.
for an (n,k) polygonation you have a similar identity,
as well as one relating faces F to vertices V
Hence

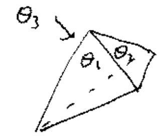
$$\chi = 2\pi(F - E + V)$$
$$= 2\pi F(1 - ? + ??) = 2$$

$\uparrow$ only 5 choices of (n,k) work; you can list them by exhaustion, and check they work.

12b Show that the torus has only two types of regular (n,k) polygonations, (although F is indeterminant). Draw a picture of each type.
[they can't be realized in $\mathbb{R}^3$ with flat polygons].

Gauss-Bonnet for paper surfaces with straight-line edges:

Recall, we defined $\displaystyle\iint_{\substack{\text{vertex}\\ \text{nbhd}}} K\,dA = 2\pi - \Theta$ "angle defecit at the vertex



$2\pi - (\theta_1+\theta_2+\theta_3)$.

(and this def made sense).

Let M be flat, except possibly at vertices.

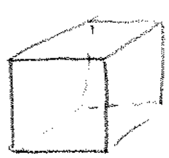
Assume M has no boundary. (could extend ~~to~~ this case.)
Then

$$\sum_{\text{vertices}} (2\pi - \Sigma\theta_i) = 2\pi\chi(M^2)$$

pf: (for a triangulation):

$$\begin{aligned} LHS &= 2\pi V - \sum_{\text{vertices}} \Sigma\theta_i \\ &= 2\pi V - \sum_{\text{triangle}} \pi \\ &= 2\pi V - \pi F \\ &= 2\pi(V-E+F) \qquad \text{since } 3F = 2E \end{aligned}$$

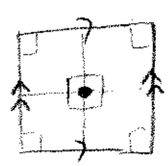
Example



LHS =

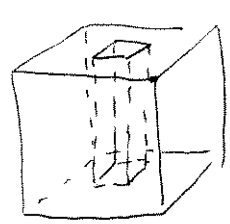
$$8\left(\tfrac{\pi}{2}\right) = 4\pi = 2\pi\cdot \underbrace{2}_{\chi(S^2)}$$

Example: flat torus:



2 vertices, no angle excess
$\Rightarrow \chi = 0$.

Example: T^2



16 vertices
for 8, $\iint K\,dA = \pi/2$!
for 8 $\iint K\,dA = -\pi/2$