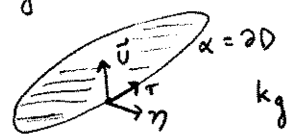
Math 4530
Monday 4/25

$\boxed{\text{local Gauss-Bonnet theorem}}$ (Will follow from Green's Theorem) $\boxed{\text{Gauss curvature bends geodesics}}$

Let D be a surface diffeomorphic to the standard unit disk $\{(u,v) \mid u^2+v^2 \le 1\}$, with smooth boundary ∂D.

Then
$$\boxed{\oint_{\partial D} k_g \, ds + \iint_D K\, dA = 2\pi}$$



$k_g = -\alpha'' \cdot \vec{\eta}$ $\quad \alpha$ pbal.

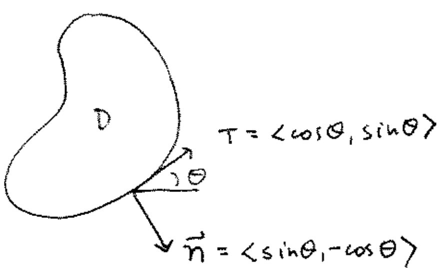
If D is a disk with piecewise smooth boundary, then

$$\boxed{\oint_{\partial D} k_g \, ds + \sum_{\text{corners}} \varepsilon_i + \iint_D K\, dA = 2\pi}$$

$\oint_{\partial D}$ means $\sum \int_{\alpha_i} k_g \, ds$

where ε_i is exterior angle at corner i; $\varepsilon_i + \theta_i = \pi$ for interior angle θ_i

Examples ① Planar domain diffeomorphic to a disk. Since $K \equiv 0$, want $\oint_{\partial D} k_g \, ds = 2\pi$.



$$\begin{aligned} k_g &= \alpha'' \cdot (-\eta) \\ &= T' \cdot (-\eta) \\ &= \langle -\sin\theta, \cos\theta \rangle \theta' \cdot \langle -\sin\theta, \cos\theta \rangle \\ &= \theta'(s) \qquad \text{(our old friend planar curvature)} \end{aligned}$$

so $\displaystyle \oint_{\partial D} k_g \, ds = \oint_{\partial D} \theta'(s)\, ds \overset{?}{=} 2\pi$

[clear for circle; for convex curve; also clear since $T(0) = T(L)$ that get a multiple of 2π]

"proof" that if D is diffeo to disk $\oint_{\partial D} \theta'(s)\, ds = 1 \cdot 2\pi$ is based on homotopy argument.

Diffeo F: unit disk $\to D$ gives a 1-parameter family of simple closed curves $\{C_r\} = \{F(S^1_r)\}$ in D, images of circles in disk.

$\displaystyle \int_{C_r} k_g(s)\, ds = \oint_{C_r} d\theta$ is continuous fcn of r, with value $2\pi k$ $\quad k \in \mathbb{Z}$.

thus k doesn't change as $r \to 0$.

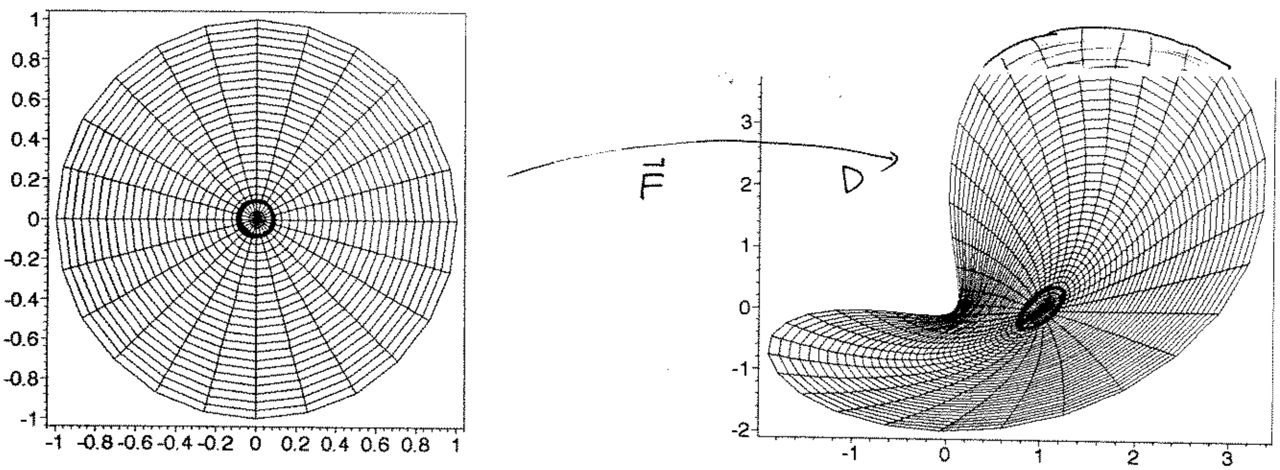
but as $r \to 0$ the curves C_r become elliptical,

so $k = 1$ as $r \to 0$. Thus $k \equiv 1$ ∎

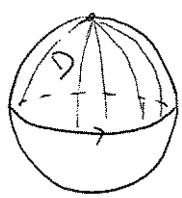
picture for page 1 proof.

$$F(u,v) \approx \underbrace{F(0,0) + \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}}_{T\binom{u}{v}} + \text{error}, \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \text{exists.}$$

image under T of S_r is an exact ellipse.



Example 2

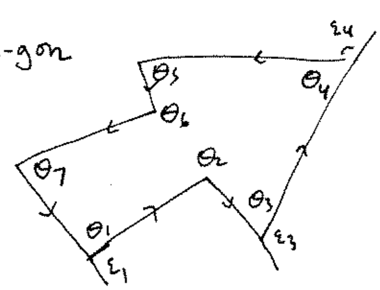


D = upper hemisphere

$$\left.\begin{aligned} &\iint_D K\,dA = \iint_D \frac{1}{R^2}\,dA = \frac{1}{R^2}\,\frac{1}{2}\,4\pi R^2 = 2\pi \\ &\int_{\partial D} k_g\,ds = 0 \end{aligned}\right\} \text{sum} = 2\pi \ \checkmark$$

Example 3 polygons in $\mathbb{R}^2$

n-gon



$k_g \equiv 0$ along edges, $K \equiv 0$

2^{nd} formula says

$$0 + \sum_i \varepsilon_i = 2\pi$$

$$\sum_i (\pi - \theta_i) = 2\pi$$

$$n\pi - \sum \theta_i = 2\pi$$

$$\boxed{(n-2)\pi = \sum \theta_i}$$

(you can prove this by elementary means)

e.g.  $\Rightarrow \sum \theta_i = \pi$

 $\Rightarrow \sum \theta_i = 2\pi$

 $\Rightarrow \sum \theta_i = 3\pi$ etc.

Example 4 geodesic triangles (sides are geodesics)

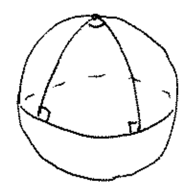
$$\sum_{3 \text{ corners}} (\pi - \theta_i) + \iint_D K\,dA = 2\pi$$

$$(\theta_1 + \theta_2 + \theta_3) = \iint_D K\,dA + \pi$$

- If $K \neq 0$ is constant
  can understand all cases from $K=1$ and $K=-1$,
  since if scale metric by R area scales by R^2
  K scales by $\frac{1}{R^2}$
  so $K \cdot$ area stays the same.

can derive these facts using elementary calculations – see Fri 4/22 notes

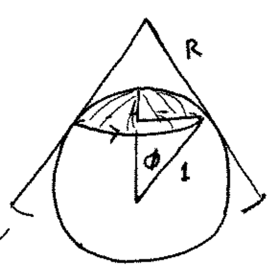
- on unit sphere
  $\theta_1 + \theta_2 + \theta_3 = \text{area} + \pi$ "fat Δ's.



- in hyperbolic space
  $\theta_1 + \theta_2 + \theta_3 = -\text{area} + \pi$
  "skinny Δ's"

Example 5 unit Sphere latitudes & polar regions

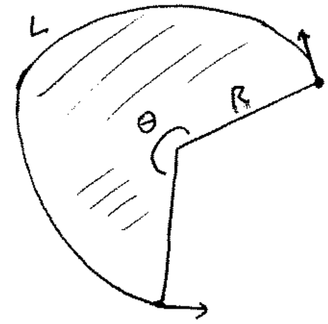
$0 < \phi < \pi/2$



cone tent.

k_g is in fact intrinsic, so unroll the cone covering the latitude of radius $\sin\phi$: its flat,

so $\boxed{\int_C k_g\,ds = \Theta}$



$$\iint K\,dA = \int_0^\phi \underbrace{2\pi\sin\alpha}_{L \text{ of latitude with angle } \alpha}\,d\alpha = 2\pi(1-\cos\phi)$$

$R = \tan\phi$

$\Theta R = L = 2\pi\sin\phi$

$\Theta \frac{\sin\phi}{\cos\phi} = 2\pi\sin\phi$

$\boxed{\Theta = 2\pi\cos\phi}$

thus $\iint_{\text{cap}} K\,dA + \int_C k_g\,ds$

$= 2\pi(1-\cos\phi) + 2\pi\cos\phi$

$= 2\pi$ ✓

Example 6 (Example 5 turned around).

<u>Cones</u> : If we rounded the cone near the vertex, its $\iint K\,dA$ = area of Gauss map image (old-old HW!)

= area of spherical cap

= $2\pi(1-\cos\phi)$

$$\boxed{\iint_{\text{cone point}} K\,dA = 2\pi - \Theta}$$

= "angle defecit" of cone

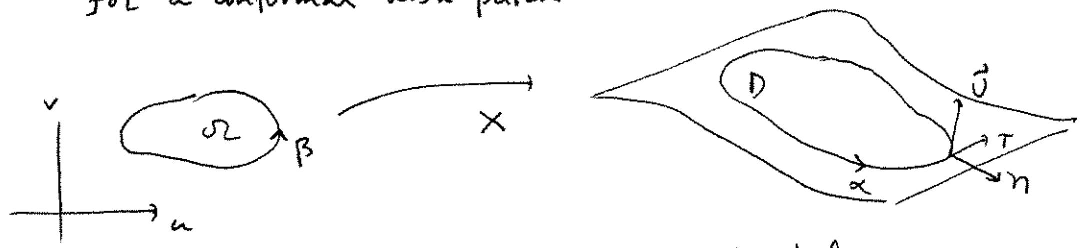
this is eactly how to define $\iint K\,dA$ at any cone point of a flat surface to make

$$\boxed{\underbrace{\oint_{\partial D} k_g\,ds}_{\Theta} + \iint K\,dA = 2\pi}$$

true, even for $\Theta > 2\pi$.

you can make lots of neat examples of surfaces which are flat except at cone points, using paper!

proof of local Gauss-Bonnet
for a conformal disk patch



α pbal

$$X_u \cdot X_u = X_v \cdot X_v = \varphi^2$$
$$X_u \cdot X_v = 0$$

in earlier HW you checked

$$K = -\frac{1}{\varphi^2}\left(\left(\frac{\varphi_u}{\varphi}\right)_u + \left(\frac{\varphi_v}{\varphi}\right)_v\right)$$

$$dA = \varphi^2\, du\, dv$$

$$\Rightarrow \iint_D K\, dA = \iint_\Omega -\left[\left(\frac{\varphi_u}{\varphi}\right)_u + \left(\frac{\varphi_v}{\varphi}\right)_v\right] du\, dv$$

$$\boxed{\iint_D K\, dA = \int_{\partial\Omega} \frac{\varphi_v}{\varphi}\, du - \left(\frac{\varphi_u}{\varphi}\right) dv}$$

Green's thm.

$$\oint_\alpha k_g\, ds = ?$$

$$k_g = \alpha'' \cdot (-\eta)$$
$$= (X_{,i}\dot\beta^i)' \cdot (-\eta)$$
$$= (\underbrace{X_{,ij}}_{\Gamma_{ij}^k X_{,k}} \dot\beta^i \dot\beta^j + X_{,i}\ddot\beta^i) \cdot (-\eta)$$

...

better to avoid Christoffel calculation -
instead write

$$\vec T = \cos\theta\left(\frac{X_u}{\varphi}\right) + \sin\theta\left(\frac{X_v}{\varphi}\right) \quad ; \quad \text{so } \theta = \measuredangle\, X_u, T$$
$$= \measuredangle\, \vec e_1, \beta' \quad \text{also, since } X \text{ is conformal}$$

$$\Rightarrow -\vec\eta = -\sin\theta\left(\frac{X_u}{\varphi}\right) + \cos\theta\left(\frac{X_v}{\varphi}\right)$$

$$k_g = \vec T' \cdot (-\vec\eta) = \left\{\frac{1}{\varphi}\left(\cos\theta\, X_u + \sin\theta\, X_v\right)\right\}' \cdot (-\vec\eta) \; ;$$

triple product rule:

$$= \left(\frac{1}{\varphi}\right)' \underbrace{(\varphi \vec T)\cdot(-\vec\eta)}_{0} + \underbrace{\frac{1}{\varphi}\theta'(-\sin\theta X_u + \cos\theta X_v)\cdot(-\vec\eta)}_{\underbrace{\theta'(-\vec\eta)\cdot(-\vec\eta)}_{\boxed{\theta'(s)}}} + \frac{1}{\varphi}\left(\cos\theta\, X_u' + \sin\theta\, X_v'\right)\cdot\left(\frac{1}{\varphi}\right)(-\sin\theta X_u + \cos\theta X_v)$$

$$\frac{1}{\varphi^2}\Big[\cos\theta\sin\theta\,\underbrace{\cancel{(-X_u\cdot X_u' + X_v\cdot X_v')}}_{(-\varphi^2)' + (\varphi^2)' = 0} + X_u'\cdot X_v\Big]$$

(since $X_v' \cdot X_u = -X_u' \cdot X_v$)

$$\frac{1}{\varphi^2}(X_u' \cdot X_v)$$
$$= \frac{1}{\varphi^2}(X_{uu}u' + X_{uv}v')\cdot X_v$$
$$= \frac{1}{\varphi^2}(-\varphi\varphi_v u' + \varphi\varphi_u v')$$
$$\boxed{= -\frac{\varphi_v}{\varphi}u' + \frac{\varphi_u}{\varphi}v'}$$

$$X_{uv}\cdot X_v = \frac{1}{2}(\varphi^2)_u = \varphi\varphi_u$$
$$X_{uu}\cdot X_v = -X_u\cdot X_{uv} = -\frac{1}{2}(\varphi^2)_v = -\varphi\varphi_v$$

$$\Rightarrow \oint_\alpha k_g\,ds = \int_0^L \frac{d\theta}{ds}\,ds + \int_0^L -\frac{\varphi_v}{\varphi}u' + \frac{\varphi_u}{\varphi}v'\,ds \qquad (*)$$

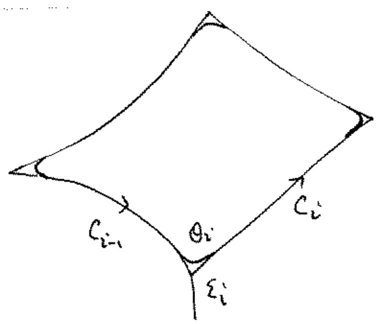
s is arclength on α, not on β, but $\beta(s)$ parameterizes $\partial\Omega$, so

$$\int_0^L \frac{d\theta}{ds}\,ds = \oint_\beta d\theta = 2\pi \text{ page 1}$$

$$\int_0^L -\frac{\varphi_v}{\varphi}u' + \frac{\varphi_u}{\varphi}v'\,ds = \oint_\beta -\frac{\varphi_v}{\varphi}\,du + \frac{\varphi_u}{\varphi}\,dv = \iint_\Omega \left(\frac{\varphi_u}{\varphi}\right)_u + \left(\frac{\varphi_v}{\varphi}\right)_v du\,dv = -\iint_D K\,dA$$

$$\Rightarrow \boxed{\oint_\alpha k_g\,ds + \iint_D K\,dA = 2\pi}$$

The case of polygonal D follows by approximation with smooth domains:



if $\alpha^\varepsilon(s)$ is the smooth approximation, for which the eqtn at the top of this page, $*$, holds,

$$\oint_{\alpha^\varepsilon} k_g\,ds \longrightarrow \underbrace{\sum_i \int_{C_i} \frac{d\theta}{ds}\,ds}_{\oint_\alpha k_g\,ds} + \sum_i \varepsilon_i + \underbrace{\sum_i \int_{C_i} -\frac{\varphi_v}{\varphi}\,du + \frac{\varphi_u}{\varphi}\,dv}_{-\iint_D K\,dA}$$