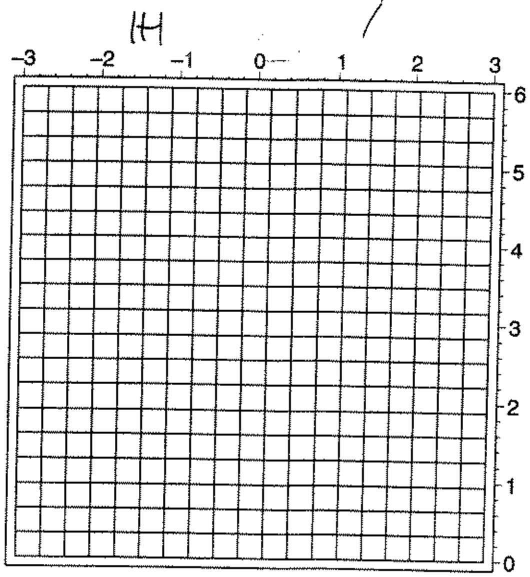
Math 4530-1
Fri 4/22

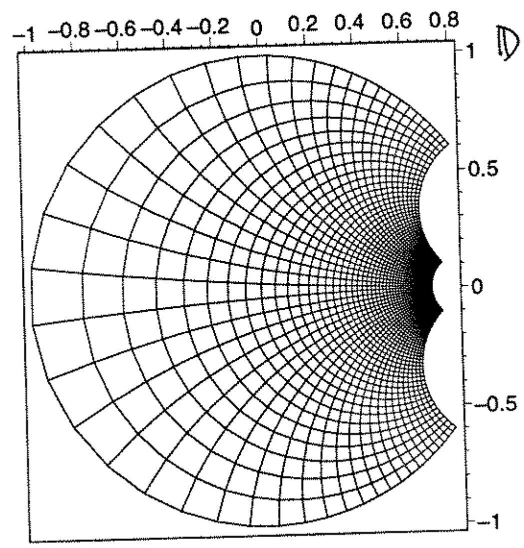
finishing our study of the hyperbolic plane

- Do proof of Thm 1, page 4 Wed.
- page 5 Wed. (ammended)

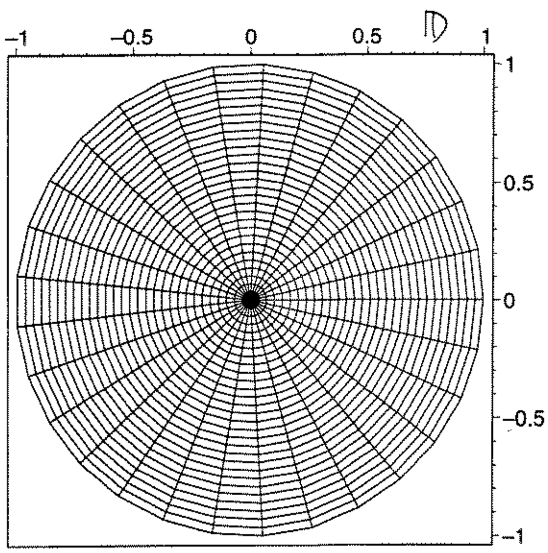
Maple's proof of the geometry of $T(z) = \frac{z-i}{z+i}$, $S(w) = \frac{iw+i}{-w+1}$:

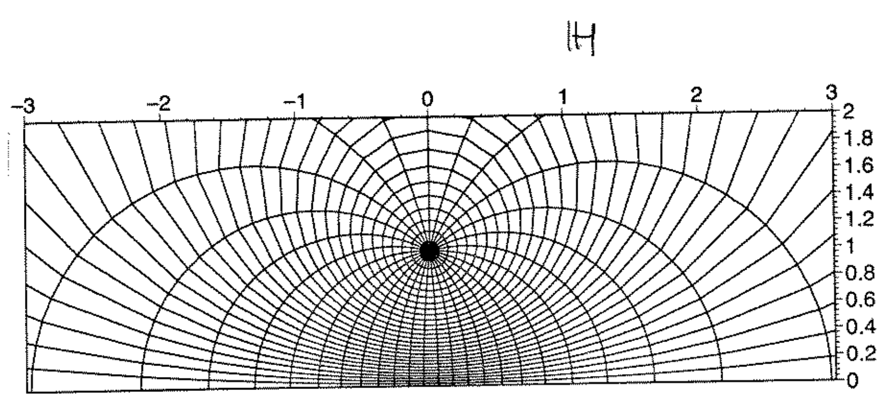
$T(z) = \frac{z-i}{z+i}$





$S(w) = \frac{iw+i}{-w+1}$





pf of isometry thm from page 5 Wed.

Using the isometry between $\mathbb{D}$ and $\mathbb{H}$, it suffices to show that every isometry of $\mathbb{D}$ is a lft, knowing that every lft of $\mathbb{D}$ is an isom!

<u>Lemma</u>: Let $F: \mathbb{D} \to \mathbb{D}$ an isometry
$F(0) = 0$
Then $F(z) = e^{i\theta} z$ is a rotation.

<u>proof</u> $F(0) = 0$

$$F \text{ isom} \Rightarrow |F_* v|_{\mathbb{H}} = |v|_{\mathbb{H}} \quad v \in T_0 \mathbb{D}^2$$

$$|F_* v|_E = |v|_E$$

$$\Rightarrow |F'(0)| = 1$$

$$\Rightarrow F'(0) = e^{i\theta}$$

Consider $G(z) = e^{-i\theta} F(z)$

Then $G'(0) = 1$

$$\Rightarrow G_* : T_0 \mathbb{D}^2 \to T_0 \mathbb{D}^2 = id$$

Let α be a geodesic $\alpha(0) = 0$ (i.e. trace of α is a segment thru 0)

then $G \circ \alpha$ is a geodesic, with

$$G \circ \alpha(0) = 0$$

$$(G \circ \alpha)'(0) = G_* \alpha'(0) = \alpha'(0)$$

$$\Rightarrow G \circ \alpha = \alpha \quad \forall \alpha$$

$$\Rightarrow G = id \quad (G(z) = z \ \forall z!)$$

$$\Rightarrow F(z) = e^{i\theta} G(z) = e^{i\theta} z. \quad \blacksquare$$

Now let $F: \mathbb{D} \to \mathbb{D}$ an isometry.
$F(0) = z_0 \in \mathbb{D}$

Let $G(z) = \dfrac{z - z_0}{1 - \bar{z}_0 z}$ Note $G(z_0) = 0$

$G: \mathbb{D} \to \mathbb{D}$: $\left| \dfrac{z - z_0}{1 - \bar{z}_0 z} \right| = \dfrac{|z - z_0|}{|\bar{z}||1 - \bar{z}_0 z|}$ $(|z| = 1)$

If $|z| = 1$ $= \dfrac{|z - z_0|}{|\bar{z} - \bar{z}_0|} = 1$

So $G \circ F$ is an isom (because G is)

$G \circ F(0) = 0$

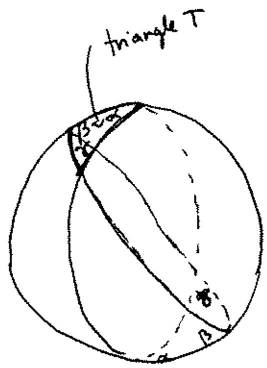
Lemma $\Rightarrow G \circ F(z) = e^{i\theta} z$

$\Rightarrow F(z) = G^{-1}(e^{i\theta} z)$ is an lft $\blacksquare$

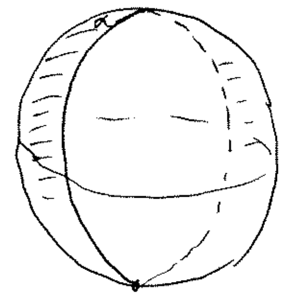
HW10 Use ideas of this proof to show that every isometry of S^2 is of the form $F(\vec{x}) = \Theta\vec{x}$ where Θ is an orthogonal matrix.

Geodesic Δ's and area (getting ready for Gauss-Bonnet thm M,W next week)

on S^2



"lune"



$$\text{area} = \left(\frac{2\alpha}{2\pi}\right)(4\pi) = 4\alpha$$

↑ area of S^2

adding areas of the α, β, γ lunes

yields $4(\alpha+\beta+\gamma)$

- Any two lunes intersect in T & its reflection
- The three lunes cover all of S^2. Except for T & its reflection, pts are only in one lune (or on ∂)

Thus

$$6|T| + (4\pi - 2|T|) = 4(\alpha+\beta+\gamma)$$

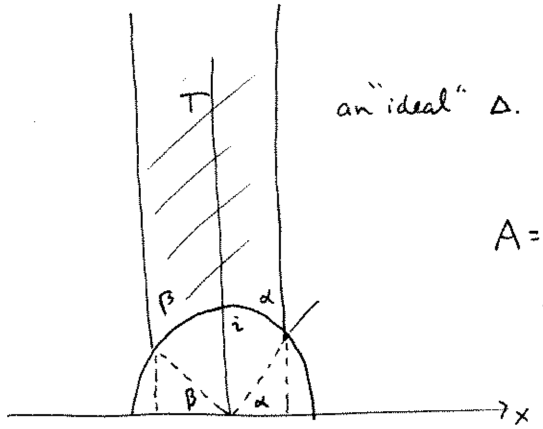
$$4|T| + 4\pi = 4(\alpha+\beta+\gamma)$$

$$\boxed{|T| = \alpha+\beta+\gamma-\pi}$$

triangles are "fat"

on $\mathbb{H}^2$

$$dA_{\mathbb{H}} = \left(\tfrac{1}{y}\,dx\right)\left(\tfrac{1}{y}\,dy\right) = \frac{1}{y^2}\,dx\,dy$$

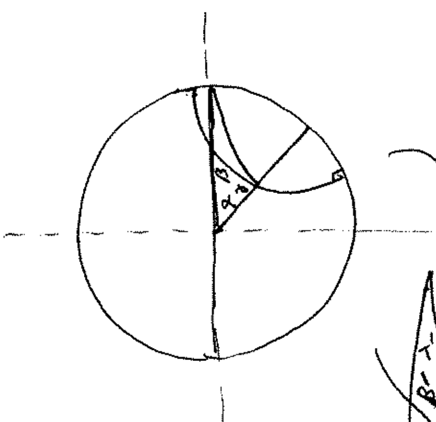


an "ideal" Δ.

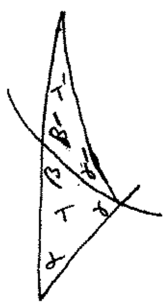
$$A = \int_{-\cos\beta}^{\cos\alpha} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}\,dy\,dx$$

HW11 Show $A = \pi - (\alpha+\beta)$

Thus, for a general geodesic Δ:



magnify.



$$|T \cup T'| = \pi - (\alpha + \gamma + \gamma')$$
$$= |T| + |T'|$$
$$|T| = \pi - (\alpha+\gamma+\gamma') - \underbrace{|T'|}_{(\pi - (\gamma' + \beta'))}$$
$$= -\alpha - \gamma + \beta'$$
$$= -\alpha - \gamma + (\pi - \beta)$$
$$\boxed{|T| = \pi - (\alpha+\beta+\gamma)}$$

triangles are "skinny"

(One needs to know that congruent ideal geodesic Δ's can be isometrically mapped to eachother with lft's, so that the HW11 formula holds for any ideal Δ).