Math 4530-1

Fri 4/22

finishing our study of the hyperbolic plane

- Do proof of Thm 1, page 4 Wed.
- page 5 Wed. (announced)

Maple's proof of the geometry of $T(z) = \frac{2 - i}{2 + i}$, $S(w) = \frac{-i - w + i}{-w + 1}$.
pf of isometry thm from page 5 Wed.

Using the isometry between $D$ and $H$, it suffices to show that every isometry of $D$ is a lift, knowing that every lift of $D$ is an isom!

**Lemma:** Let $F: D \to D$ be an isometry

- $F(0) = 0$
- Then $F(z) = e^{i\theta}z$ is a rotation.

**Proof**

$F(0) = 0$

$F$ isom $\Rightarrow |F_v vl| = |vl|_{\mathbb{E}} \quad \forall v \in T_0 D$

$\therefore |F_v vl|_E = |vl|_E$

$\Rightarrow |F'(0)| = 1$

$\Rightarrow F'(0) = e^{i\theta}$

Consider $G(z) = e^{i\theta}F(z)$

Then $G'(0) = 1$

$\Rightarrow G: T_0 D \to T_0 D = id$

Let $\alpha$ be a geodesic $\alpha(0) = 0$ (i.e. trace $\alpha \times$ is a segment thru 0)

Then $G \alpha$ is a geodesic, with

- $G \alpha(0) = 0$
- $(G \alpha')(0) = G \alpha'(0) = \alpha'(0)$

$\Rightarrow G \alpha = \alpha \quad \forall \alpha$

$\Rightarrow G = id.$

$\Rightarrow G(z) = z \quad \forall z!$

$\Rightarrow F(z) = e^{i\theta}G(z) = e^{i\theta}z.$

Now let $F: D \to D$ be an isometry.

- $F(0) = z_0 \in D$

Let $G(z) = \frac{z - z_0}{1 - z_0 \bar{z}}$

Note $G(z_0) = 0$

So $G \circ F$ is an isom

(because $G$ is)

$G \circ F(0) = 0$

**Lemma**

$\Rightarrow G \circ F(z) = e^{i\theta}z$

$\Rightarrow F(z) = G^{-1}(e^{i\theta}z)$ is an lift
Use ideas of this proof to show that every isometry of $S^2$ is of the form $F(x) = \Theta x$ where $\Theta$ is an orthogonal matrix.

Geodesic $\Delta$'s and area (getting ready for Gauss-Bonnet this HW next week)

![Diagram of a sphere with labeled angles and a lune]

Adding areas of the $\alpha, \beta, \gamma$ lunes yields $4(\alpha + \beta + \gamma)$.

- Any two lunes intersect in $T$ & its reflection.
- The three lunes cover all of $S^2$. Except for $T$ & its reflection, pts are only in one lune.

Thus

$$6|T| + (4\pi - 2|T|)$$

$$= 4(\alpha + \beta + \gamma)$$

$$4|T| + 4\pi = 4(\alpha + \beta + \gamma)$$

$$|T| = \alpha + \beta + \gamma - \pi$$

Triangles are "fat".
\[ dA_{\Delta} = \left( \frac{1}{y^2} \right) \left( \frac{1}{y^2} \right) dx \\, dy \]

Thus, for a general geodesic \( \Delta \):

\[ A = \int_{-\cos \beta}^{\cos \alpha} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2} \, dy \, dx \]

(Hint) Show \( A = \pi - (\alpha + \beta) \)

\[ \text{ITU'1} = \pi - (\alpha + \beta + \beta') \]
\[ = \text{IT1} + \text{IT'1} \]
\[ \text{IT1} = \pi - (\alpha + \beta + \beta') = (\pi - (\alpha + \beta')) \]
\[ = -\alpha + \beta' \]
\[ = -\alpha + \gamma + (\pi - \beta) \]

\[ \text{IT1} = \pi - (\alpha + \beta + \beta') \]

(One needs to know that congruent geodesic \( \Delta \)'s can be isometrically mapped to each other with left's, so that the formula holds for any ideal \( \Delta \).)