Math 4530
Wed 4/20

Last HW assignment
(which we are still accumulating)
is <u>optional</u> for people who are doing a project in lieu of final exam.
For everyone else, it will constitute a "take-home" portion of your final exam grade, (= 40% of total).
The completed assignment will be due Wed 5/3, 10:30 a.m. at start of in-class final.

Geodesics and isometries of the hyperbolic plane

Recall def. of <u>isometry</u> $F: M^2 \to N^2$
- diffeo
- $F_*: T_pM \to T_{F(p)}N$ isom $\forall p$

<u>Hyperbolic plane</u> $\mathbb{H} = \{(x,y) : y>0\}$

metric $[g_{ij}] = \begin{bmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{bmatrix}$

i.e. if $\vec{v}, \vec{w} \in T_pH$, $\vec{v}\cdot\vec{w}_{\mathbb{H}} := \frac{1}{y^2}\vec{v}\cdot\vec{w}_{\mathbb{E}}$; $ds_{\mathbb{H}} = \frac{1}{y} ds_{\mathbb{E}}$

<u>Poincaré disk</u> $\mathbb{D} = \{(u,v) : u^2+v^2<1\}$

metric $[g_{ij}] = \frac{4}{(1-(u^2+v^2))^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $ds_{\mathbb{D}} = \frac{2}{1-(u^2+v^2)} ds_{\mathbb{E}}$

<u>Theorem 1</u>: Write $z = x+iy$
$w = u+iv$

Then $T(z) = \frac{z-i}{z+i}$ is an isometry from $\mathbb{H}$ to $\mathbb{D}$

(with inverse isometry $S(w) = \frac{iw+i}{-w+1} = \frac{w+1}{iw-i}$)

We'll discuss this below...

Theorem 2 Isometries preserve geodesics (see ODE!)

Theorem 3 Let $F: M \to M$ be an isometry s.t. a curve $\alpha(s)$ is the fixed point set of F

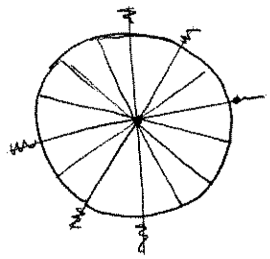
i.e. $F(x) = x$ iff $x = \alpha(s)$ some s.

Then α is a geodesic.

examples lines in $\mathbb{R}^2$ are fixed by reflections across the line

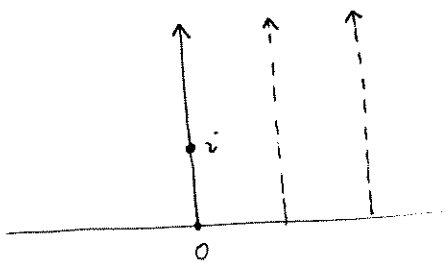
great circles on S^2 are fixed by reflections through the planes containing them

$\mathbb{D}^2$:



segments thru origin of $\mathbb{D}^2$ are geodesics, because they are the fixed pts of Euclidean reflections across lines thru origin. Since such reflections preserve distance to origin they preserve $ds_{\mathbb{D}}$, and so are isometries of $\mathbb{D}^2$.

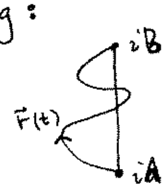
$\mathbb{H}^2$



the positive y-axis is a geodesic because it is preserved by the $\mathbb{H}^2$ isometry $(x,y) \mapsto (-x,y)$.

(all half rays $\begin{matrix} x = a \\ y > 0 \end{matrix}$ also geodesics because ~~preserved~~ they are the image of positive y-axis under horizontal translations

(in fact any arc of this geodesic is length minimizing:



$$L(\vec{r}) = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; \frac{1}{y(t)}\, dt$$

$$\geq \int_a^b \left|\frac{dy}{dt}\right| \frac{1}{y}\, dt$$

$$\geq \int_a^b \frac{\frac{dy}{dt}}{y}\, dt = \ln y(t)\Big]_a^b = \ln(B/A)$$

$$= \int_A^B \frac{dy}{y} = \text{hyperbolic length of segment!} \Big)$$

In fact $\alpha(s) = i e^s \quad -\infty < s < \infty$ is pbal!

proof of Thm 3

Recall the geodesic ODE system

$$\begin{cases} \ddot{\beta}^k + \Gamma^k_{ij}\, \dot{\beta}^i \dot{\beta}^j = 0 & k = 1,2 \\ \beta^k(0) = q^k & \text{''} \\ \dot{\beta}^k(0) = v_0^k & \text{''} \end{cases}$$

which has unique soltns.

Solve this system with

$\beta(0) = \alpha(0) = q$ (technically $\beta(0) = X^{-1}(\alpha(0))$)

$\dot{\beta}(0) = \alpha'(0)$.

$F \circ \alpha = \alpha \Rightarrow F_*(\alpha'(0)) = \alpha'(0)$

$\Rightarrow F \circ \beta$ is geodesic for same IVP! $\Rightarrow \beta$ is fixed by F, by uniqueness!

$\Rightarrow \beta = \alpha$ ∎

Digression to linear fractional transformations of $\mathbb{C} \cup \{\infty\}$

$$F(z) = \frac{az+b}{cz+d} \qquad \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If $$G(w) = \frac{ew+f}{gw+h} \qquad \sim \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Then $G(F(z))$ is a l.f.t., with matrix $\begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$!

check it!

Cor: If $F \sim A$, $\det A \neq 0$, then $F^{-1} \sim A^{-1}$

note, you can scale A without changing F, so can assume $\det A = +1$ or -1

($\det A = 0 \Rightarrow F$ const, don't consider).

Example: If $T(z) = \frac{z-i}{z+i}$ show $T^{-1}(w) = \frac{iw+i}{-w+1}$. (c.f. page 1)

Theorem: lft's transform {lines, circles} into {lines, circles}.

proof: This is a corollary of the fact that any lft is a composition of translations, inversions, scalings:

HW 5 Prove that if $T(z) = \frac{az+b}{cz+d}$ then (for $c \neq 0$), $T = T_4 \circ T_3 \circ T_2 \circ T_1$

$$T_1(z) = z + \frac{d}{c}$$
$$T_2(z) = \frac{1}{z}$$
$$T_3(z) = \frac{bc-ad}{c^2} z$$
$$T_4(z) = z + \frac{a}{c}$$

thus the lft thm holds if we can show that inversion preserves the class of lines and circles

HW6 (hard!) Show $T(z) = \frac{1}{z}$ maps $\{\text{lines, circles}\} \to \{\text{lines, circles}\}$.
(hints may follow).

Proof of thm 1 : Consider $T(z) = \frac{z-i}{z+i}$

If x is real then $T(x) = \frac{x-i}{x+i}$; $|T(x)| = \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} = 1$.

So T maps the real line to the unit circle.

Since $T(i)=0$ it maps $\mathbb{H}$ to $\mathbb{D}$. $\left(\text{or } T(x+iy) = \frac{x+iy-i}{x+iy+i} = \frac{x+i(y-1)}{x+i(y+1)} \right.$

$\left. \text{so } |T(z)|^2 = \frac{x^2+(y-1)^2}{x^2+(y+1)^2} \right)$

isometry condition:

recall, for a conformal map, $T(z)$, the conformal scaling factor is $|T'(z)|$.

We need to check if

$$|T_* v|_{\mathbb{D}^2} = |v|_{\mathbb{H}^2} \quad \text{if } v \in T_{x+iy}\mathbb{H}^2.$$

$$|v|_{\mathbb{H}^2} = \frac{|v|_{\mathbb{E}^2}}{y}$$

$$|T_* v|_{\mathbb{D}^2} = |T'(x+iy)| \frac{2}{1-|T(z)|^2} |\vec{v}|_{\mathbb{E}}$$

$$T'(z) = \frac{z+i-(z-i)}{(z+i)^2} = \frac{2i}{(z+i)^2} =$$

$$\text{So } |T'(z)| = \frac{2}{|z+i|^2} = \frac{2}{|x+iy+i|^2} = \frac{2}{x^2+(y+1)^2}$$

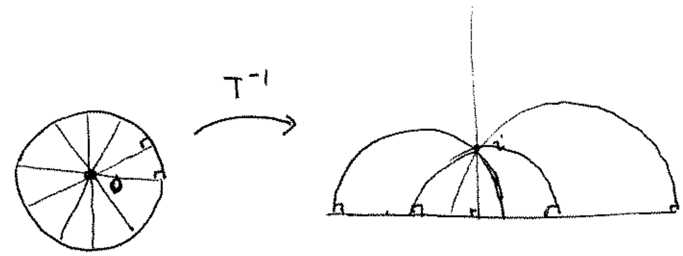
$$\text{So } |T_* v|_{\mathbb{D}^2} = \frac{2}{x^2+(y+1)^2} \frac{2}{1-\left[\frac{x^2+(y-1)^2}{x^2+(y+1)^2}\right]} |\vec{v}|_{\mathbb{E}^2}$$

$$= \frac{4}{x^2+(y+1)^2 - (x^2+(y-1)^2)} |\vec{v}|_{\mathbb{E}^2}$$

$$= \frac{1}{y} |\vec{v}|_{\mathbb{E}^2} \quad !! \quad \blacksquare$$

<u>Corollary</u> : The geodesics thru i in $\mathbb{H}^2$ are half circles meeting the real axis $\perp$'ly (& the imag axis)

<u>pf</u> :



conformal $\Rightarrow$ preserves angles
lft $\Rightarrow$ circles, lines $\to$ circles lines.

<u>Theorem</u> The orientation preserving isometries of $\mathbb{H}$ are exactly the lft's $F(z) = \frac{az+b}{cz+d}$ with $a,b,c,d \in \mathbb{R}$, $ad-bc = 1$ (i.e. $ad-bc > 0$ before scaling the matrix) i.e. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{R})$

<u>proof</u> :

HW7 $T_1(z) = az$ $a > 0$ is an isometry of $\mathbb{H}^2$
$T_2(z) = z + b$ b real " " "
$T_3(z) = -\frac{1}{z}$ " " " " "

HW8 Every f as above can be written as a suitable composition of the T_i

Thus the F's above <u>are</u> all isometries of $\mathbb{H}$.

HW9 the only lft's which map $\mathbb{H}$ to $\mathbb{H}$ are ones which can be written as

$$F(z) = \frac{az+b}{cz+d} \quad \text{with } a,b,c,d \in \mathbb{R}, \quad ad-bc = 1$$

hint $F(x) \in \mathbb{R}\ \forall x \in \mathbb{R} \to$ so e.g. $\frac{b}{d} \in \mathbb{R}$, $\frac{a}{c} \in \mathbb{R}$...
$F'(x) > 0\ \forall x \in \mathbb{R}$ since F preserves orientation
($F'(x) = F_* \vec{e}_1$; if $F'(x) < 0$ $F_* \vec{e}_2$ points down)

We can now prove the isometry thm...

reduces to showing that every orientation preserving isometry of $\mathbb{H}^2$ is a lft, knowing already that every lft: $\mathbb{H}^2 \to \mathbb{H}^2$ <u>is</u> an isometry