Math 4530
Monday 4/18

Recall, α is geodesic iff $\frac{d}{dt} L(\alpha^t)\Big|_{t=0} = 0 \quad \forall$ 1-parameter variations of α fixing endpoints

$$\iff \alpha''(s) = \kappa\vec{N} \,/\!/\, \vec{U}(\alpha(s)) \quad \text{when } \alpha \text{ pbal} \qquad \text{(extrinsic description)}$$

$$\iff \ddot{\beta}^k + \Gamma_{ij}^k \dot{\beta}^i \dot{\beta}^j = 0, \quad k=1,2 \quad \text{for } \alpha = X\circ\beta, \quad X: D \to M^2 \text{ a patch.}$$

(use "$\cdot$" for $\frac{d}{ds}$)

$\to$ page 3 Fri.

$\to$ page 3-4 shows an intrinsic derivation of these differential eqtns.

What does this system look like for orthogonal (and conformal) coordinates?

$\beta(s) = \langle u(s), v(s) \rangle$

from page 136: $[g_{ij}] = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}$

$\Gamma_{11}^1 = \frac{E_u}{2E}$

$\Gamma_{11}^2 = -\frac{E_v}{2G}$

$\Gamma_{12}^1 = \Gamma_{21}^1 = \frac{E_v}{2E}$

$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{G_u}{2G}$

$\Gamma_{22}^1 = -\frac{G_u}{2E}$

$\Gamma_{22}^2 = \frac{G_v}{2G}$

so if $[g_{ij}] = \begin{bmatrix} \varphi^2 & 0 \\ 0 & \varphi^2 \end{bmatrix}$

$\Gamma_{11}^1 = \frac{\varphi_u}{\varphi}$

$\Gamma_{11}^2 = -\frac{\varphi_v}{\varphi}$

$\Gamma_{12}^1 = \Gamma_{21}^1 = \frac{\varphi_v}{\varphi}$

$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{\varphi_u}{\varphi}$

$\Gamma_{22}^1 = -\frac{\varphi_u}{\varphi}$

$\Gamma_{22}^2 = \frac{\varphi_v}{\varphi}$

So:

$$\boxed{\begin{aligned} &\ddot{u}(s) + \left(\frac{E_u}{2E}\right)(\dot{u})^2 + 2\left(\frac{E_v}{2E}\right)\dot{u}\dot{v} - \frac{G_u}{2E}\dot{v}^2 = 0 \\ &\ddot{v} - \left(\frac{E_v}{2G}\right)(\dot{u})^2 + 2\left(\frac{G_u}{2G}\right)\dot{u}\dot{v} + \left(\frac{G_v}{2G}\right)\dot{v}^2 = 0 \end{aligned}}$$

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$$\boxed{\begin{aligned} &\ddot{u} + \frac{\varphi_u}{\varphi}(\dot{u})^2 + 2\frac{\varphi_v}{\varphi}\dot{u}\dot{v} - \frac{\varphi_u}{\varphi}\dot{v}^2 = 0 \\ &\ddot{v} - \frac{\varphi_v}{\varphi}(\dot{u})^2 + 2\frac{\varphi_u}{\varphi}\dot{u}\dot{v} + \frac{\varphi_v}{\varphi}\dot{v}^2 = 0 \end{aligned}}$$

(Exercise not to hand in: check these!)

Examples

- sphere (great circles; did)
- plane (straight lines; did)
- cylinder or cone → see last page of Fri notes

geodesics are an isometry invariant, because

→ the DE for β only depends on the Γ^i_{jk}'s, which only depend on $[g_{ij}]$ & derivs

or

→ First variation of arclength $= 0$ only depends on curve lengths.

- hyperbolic space: We'll be working this out!
- surfaces of revolution: (e.g. wine bottle!)

$X(u,v) = \langle x(u), y(u)\cos v, y(u)\sin v\rangle$; $\langle x(s), y(s)\rangle$ pbal

$$[g_{ij}] = \begin{bmatrix} 1 & 0 \\ 0 & y^2(u) \end{bmatrix}$$

Let $\beta(s) = \langle u(s), v(s)\rangle$ s.t. $\alpha(s) = X\circ\beta$ is a geodesic pbal

$\alpha' = X_u u' + X_v v'$

$$\boxed{\begin{aligned} &\ddot{u} - y y_u (\dot{v})^2 = 0 \\ &\ddot{v} + 2\frac{y_u}{y}\dot{u}\dot{v} = 0 \\ &(\dot{u})^2 + y^2(\dot{v})^2 = 1 \end{aligned}}$$

(α pbal)

this 2 was missing in Fri notes

- meridians ($v = $ const) are all geodesics
[also, from extrinsic condition, this is clear, and that parallels are only geodesics when $y'(s) = 0$]

- A metric is "u-Clairaut" when $[g_{ij}]$ only depends on u. In this case the u-coord curves are always geodesics, because we may set $v = $ const, and solve the $\ddot{u}$ DE:

$$\ddot{u} + \left(\frac{E_u}{2E}\right)(\dot{u})^2 = 0$$

$$\frac{\ddot{u}}{(\dot{u})} = -\frac{E_u}{2E}\dot{u}$$

$$\ln(\dot{u}(s)) = -\frac{1}{2}\ln(E(u(s)))$$

$\dot{u}(s) = E(u(s))^{-1/2}$

which you can solve

Similarly, if a metric is v-Clairaut, you can show that the v-coord curves are geodesics.

surfaces of revolution cont'd. If v is not const,

$$\ddot{v} + \frac{2y_u}{y}\dot{u}\dot{v} = 0$$

$$\int \quad \frac{\ddot{v}}{\dot{v}} = -\frac{2y_u}{y}\dot{u}$$

$$\ln(\dot{v}(s)) = -2\ln(y(u(s))) + C \qquad (\text{assuming } \dot{v}(s) > 0$$

$$\dot{v}(s) = \frac{\tilde{C}}{y(u(s))^2}$$

recalling that $v = \theta$ on the surface of revolution,

$$\theta'(s) = \frac{\tilde{C}}{y^2} \qquad \leftarrow \theta' \text{ shrinks as } y \text{ grows (see wine bottle)}$$

Also $(\dot{u})^2 + y^2(\dot{v})^2 = 1$

$$\Rightarrow (\dot{u})^2 + y^2\frac{\tilde{C}^2}{y^4} = 1$$

$$\Rightarrow (\dot{u})^2 + \frac{\tilde{C}^2}{y^2} = 1$$

shows $|y| \geq \tilde{C}$ on the geodesic. This can cause a geodesic to "turn around"!

Finish page 5 Friday, especially the $\exists!$ thm for geodesics on the bottom of the page.

HW 1 If β solves the geodesic system on the bottom of page 5 Fri, use the chain rule to show

$$\frac{d}{dt}\left(g_{ij}(\beta(t))\,\dot{\beta}^i(t)\,\dot{\beta}^j(t)\right) \equiv 0$$

So that if $v_0^i v_0^j g_{ij}(q) = 1$, $\quad \alpha = X \circ \beta$ is pbal!

Hint: You will need the Christoffel symbol formula

$$\Gamma_{ij}^k = \frac{1}{2}g^{k\ell}\left(g_{\ell i,j} + g_{\ell j,i} - g_{ij,\ell}\right)$$

(HW1 might seem difficult)

HW2 Recall for orthogonal coords,

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u \right)$$

Show that for conformal coords $[g_{ij}] = \begin{bmatrix} \varphi^2 & 0 \\ 0 & \varphi^2 \end{bmatrix}$

$$K = -\frac{1}{\varphi^2}\left(\left(\frac{\varphi_u}{\varphi}\right)_u + \left(\frac{\varphi_v}{\varphi}\right)_v \right)$$

(should be easy)

<u>Hyperbolic plane $\mathbb{H}$</u> $= \{(x,y) : y > 0\}$

with metric $[g_{ij}] = \begin{bmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{bmatrix}$

i.e. $\varphi = \frac{1}{y}$

$$ds_{\mathbb{H}} = \frac{1}{y}\, ds_{\mathbb{R}^2} = \frac{1}{y}\sqrt{dx^2 + dy^2}$$

element of arclength in hyperbolic space

Euclidean element of arclength

HW3 Show the Gauss curvature of the hyperbolic plane is -1 (should be easy)

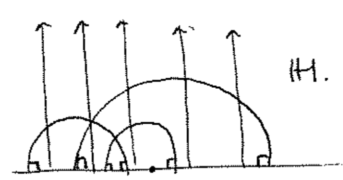
HW4 Show (starting from box on page 1), that the geodesic eqns in $\mathbb{H}$ are (for $\alpha(s) = \langle x(s), y(s) \rangle$)

4a

$$\ddot{x} - \frac{2}{y}\dot{x}\dot{y} = 0$$

$$\ddot{y} + \frac{1}{y}(\dot{x})^2 - \frac{1}{y}(\dot{y})^2 = 0$$

(should be easy)



Notice, since $\varphi = \varphi(y)$ this metric is y-Clairaut and the y-coord curves are geodesics

If $\dot{x}(s) \neq 0$, notice the 1st geodesic eqtn can be rewritten

$$\frac{\ddot{x}}{\dot{x}} = 2\frac{\dot{y}}{y} \Rightarrow \ln(\dot{x}) = 2\ln(y) + C$$

$$\Rightarrow \boxed{\dot{x} = \tilde{C} y^2}$$

But $(\dot{x})^2 + (\dot{y})^2 = y^2$ (α pbal!)

$$\Rightarrow 1 + \left(\frac{\dot{y}}{\dot{x}}\right)^2 = \frac{y^2}{(\dot{x})^2} = \frac{1}{\tilde{C}^2 y^2}$$

$$\Rightarrow \boxed{1 + \left(\frac{dy}{dx}\right)^2 = \frac{1}{\tilde{C}^2 y^2}}$$

4b Solve for $\frac{dy}{dx}$, then separate variables to show $y(x)$ is a circular arc with center on the x-axis

(see page 248 for hints)