Geodesics & geodesic curvature & physics

Let $\alpha: I \to M$, a path.
When is $\alpha$ a critical point of total length? (this is when $\alpha$ is called a geodesic)

i.e. if $\alpha^t(s) = $, $-\infty < t < \infty$
1-param. family of curves to $M$ s.t.

1) $\alpha^0 = \alpha$
2) $\alpha^t(0) = \alpha(0)$
   $\alpha^t(L) = \alpha(L)$

For what $\alpha$ is

$\frac{d}{dt} (\text{length}(\alpha^t)) |_{t=0} = 0$ for such families?

This is an intrinsic question, i.e. only depends on 1st fundamental form,
but answer is easier to visualize in the extrinsic surface setting.

Answer to question:

$\alpha^t = (\alpha^t \cdot \eta) \eta + (\alpha^t \cdot T) T + (\alpha^t \cdot U) U$

$\therefore = -k_g \eta + k_n U$

normal curvature $k_n = \alpha'' \cdot U$
geodesic curvature $k_g = -\alpha'' \cdot \eta$

$\alpha$ is a geodesic iff $k_g = 0$,
i.e. $\alpha''(s) \parallel U(\alpha(s))$

You can “see” this
**Example**

Planar curve

\[
T' = (k_g)(-\pi)
\]

and

\[
T = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}
\]

\[
T' = \frac{d\theta}{ds}
\]

\[
k_g = \frac{d\theta}{ds} = \text{planar curvature.}
\]

\[
k_\theta = 0 \iff \alpha \text{ is a line}
\]

So, \( k_g \) is a generalization of planar curvature to surfaces, and geodesics generalize straight lines.

**Extrinsic derivation of geodesic condition**

\[
\alpha^t(s) = X(\beta(s) + t \bar{u}(s))
\]

\[
\alpha^0(s) = \alpha(s) \quad \text{parallel}
\]

Expand in \( t \):

\[
\alpha^t(s) = X\left[\beta(s) + t \bar{u}(s)\right]
\]

\[
\alpha^t(s) = X\left(\beta(s) + t \bar{u}(s)\right)
\]

\[
= \alpha(s) + t Z(s) + O(t^3)
\]

Where \( Z(s) \) is a vector field along \( \alpha \), \( Z(s) \in T_{\alpha(s)} M \)

\[
L(\alpha^t) = \int_0^L 1 + t \bar{u}' Z' + O(t^3) \, ds
\]

\[
= L + t \int_0^L \alpha'(s) Z'(s) \, ds + O(t^3)
\]

Thus, for fixed endpoint deformations (\( \bar{w}(0) = \bar{z}(0) = \bar{w}(L) = \bar{z}(L) = 0 \)),

\[
\frac{d}{dt} L(\alpha^t) = -\int_0^L \alpha'' \cdot Z \, ds
\]

\[
t = 0 \quad \text{this term is zero when} \quad Z(0) = Z(L) = 0
\]

\[
1 \iff k_g = 0, \quad \alpha'' \perp T_{\alpha(s)} M \implies \alpha'' \cdot Z = 0
\]

\[
\frac{d}{dt} L(\alpha^t) = 0 \quad \forall Z \text{ tangential,}
\]

\[
t = 0 \quad \text{then} \quad \alpha'' \perp T_{\alpha(s)} M \forall s
\]

\[
i.e. \quad k_g = 0
\]
Intrinsic eqns for geodesic

\[ \alpha = X_0 \beta \]

\[ \alpha' = X_{x^i} \beta^i \]

\[ \alpha'' = X_{x^i} (\beta^{i'} \beta^i) + X_{\beta^i} \beta^i \]

\[ = (T^{ij}_{x^i} + h_{ij}) \beta^i \beta^j + X_{\beta^i} \beta^i \]

\[ \alpha'' \parallel U \iff \beta^k + T^{ij}_{x^i} \beta^i \beta^j = 0 \quad k = 1, 2 \]

For your reading pleasure:
completely intrinsic derivation of geodesic eqns: (i.e., only using [g_{ij}])

\[ \alpha^t = X (\beta + t \dot{v} \beta (s)) \]

\[ L(\alpha^t) = \int_0^L \sqrt{g_{ij}(\dot{\beta}^i + t \ddot{v} \beta (s)) (\dot{\beta}^j + t \ddot{v} \beta (s))} \, ds \]

= \int_0^L \sqrt{(g_{ij} + t g_{ij, k} \beta^k \beta^j + o(t^2)) (\dot{\beta}^i + t \ddot{v} \beta (s)) (\dot{\beta}^j + t \ddot{v} \beta (s))} \, ds \]

\[ \approx \int_0^L \sqrt{g_{ij} \dot{\beta}^i \dot{\beta}^j + t (g_{ij, k} \dot{\beta}^i \dot{\beta}^j + g_{ij} \ddot{v} \dot{\beta}^i + g_{ij} \dot{\beta}^i \ddot{v} \beta^j) + o(t^2)} \, ds \]

\[ = \int_0^L 1 + \frac{1}{2} t (\dot{v}^2) + o(t^2) \, ds \]

\[ \frac{d}{dt} L(\alpha^t) \bigg|_{t=0} = \int_0^L \frac{1}{2} (g_{ij, k} \dot{\beta}^k \dot{\beta}^i + g_{ij} \ddot{v} \dot{\beta}^i + g_{ij} \dot{\beta}^i \ddot{v} \beta^j) \, ds \]

\[ = \int_0^L \frac{1}{2} \left[ g_{ij, k} \dot{\beta}^k \dot{\beta}^i - \ddot{v} (g_{ij} \dot{\beta}^i) - \dot{v} (g_{ij} \ddot{v} \beta^j) \right] ds \]

\[ \text{(integ. by parts)} \]

\[ = \int_0^L - \ddot{v} k \left[ g_{ij} \dot{\beta}^j + \frac{1}{2} (g_{ijk} \dot{\beta}^i \beta^j + g_{ijr} \dot{\beta}^i \beta^j + g_{ikr} \dot{\beta}^i \beta^j) \right] ds \]
if $\alpha$ is stationary $\forall V$,

$$g_{ij} \dddot{\bar{p}}^j + \frac{1}{2} \left( -g_{ij,k} + g_{ij,r} \dddot{\bar{p}}^r \right) \dddot{\bar{p}}^j = 0 \quad k=1,2$$

$g^{il}$ (7c):

$$\dddot{p}^2 + T^l_{ij} \dddot{p}^i \dddot{p}^j = 0 \quad l=1,2$$

Yipes!

[one should find a better way to do intrinsic Calculus
~ "covariant derivative"
]

(next lecture!)

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**Examples**

sphere: great circles satisfy $\alpha'' || U$

every geodesic is a great circle:

$a$ parallel $\alpha''/U/\alpha \Rightarrow (\alpha \times \alpha')' = 0$

$\Rightarrow \alpha \times \alpha' = c$ (if $c \neq 0$)

$\Rightarrow (\alpha \times c)' = 0$

$\Rightarrow \alpha$ lies on plane thru origin!

hyperbolic space: ?? (good question)

cylinder \{ wrap up paper

cone \{ surfaces of revolution

$$X(u,v) = \begin{bmatrix} x(u) \\ y(u) \cos v \\ y(u) \sin v \end{bmatrix} \quad \left[ \begin{array}{c} x \\ y \\ z \end{array} \right] \text{ parallel}$$

$$[g_{ij}] = \begin{bmatrix} 1 & 0 \\ 0 & y^2(u) \end{bmatrix} \rightarrow T^j_{ik}$$

$$\left\{ \begin{array}{l} u'' - (yg_u)(v')^2 = 0 \\ V'' + \left( \frac{2}{y} \right) u'v' = 0 \\ (u')^2 + y^2 (v')^2 = 1 \end{array} \right.$$
A particle constrained to lie on a surface (but with no other external forces), satisfies

$$\dddot{x}(t) = f(t) \dot{r}(t)$$

**Newton!**

**This is geodesic again!**

notice if $\|\dot{x}(0)\| = 1$

then $\ddot{d}(\dot{x}, \dot{x}') = 2\dot{x} \cdot \dot{x}'' = 0$, so $d$ partial.

Then, light is a particle. ($E = mc^2$)

\[ \text{curved space (time)} \]

\[ \text{light} \]

\[ \text{star or black hole} \]

how many geodesics are there?

physics intuition: you should be able to start at any point and go in any direction.

math proof:

Consider the 1st order system IVP

$$\begin{cases} 
\ddot{x}^k = v^k & k = 1, 2 \\
\dot{v}^k + T^k_{\;ij}v^iv^j = 0 & k = 1, 2 \\
\dot{x}(0) = x_0 \\
\dot{v}(0) = v_0 
\end{cases}$$

**Existence solution**, and in fact $x = X_{\dot{\alpha}}$ is partial, hence satisfies geodesic again.

check $\dddot{d}(g_{ij}, \dot{x}, \ddot{x}) = 0$ from the system.