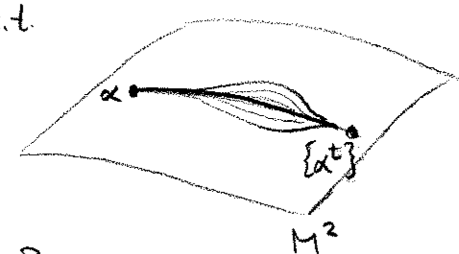
Friday April 1
Math 4530

Geodesics & geodesic curvature & physics

Let $\alpha : I \to M$, α pbal.
When is α critical wrt total length? (this is when α is called a geodesic)

i.e. if $\alpha^t(s)$ $\quad -\varepsilon < t < \varepsilon$
1-par. family of curves to M s.t.

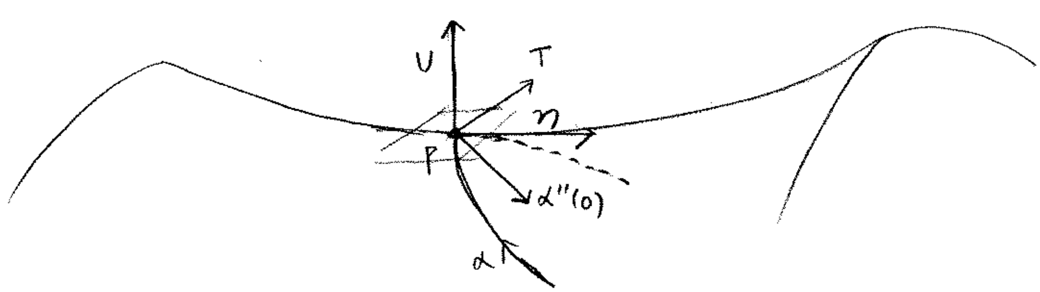
(1) $\alpha^0 = \alpha$
(2) $\alpha^t(0) = \alpha(0)$
$\alpha^t(L) = \alpha(L)$



For what α is
$$\frac{d}{dt}\left(\text{length}(\alpha^t)\right)\Big|_{t=0} = 0 \quad \forall \text{ such families?}$$

This is an intrinsic question, i.e. only depends on 1st fundamental form, but answer is easier to visualize in the extrinsic surface setting.

Answer to question:



$\alpha : I \to M$
α pbal
$\alpha(0) = p$
$\alpha'(0) = T$
$\eta := T \times U$
↑
conormal

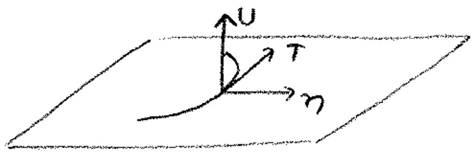
$$\alpha'' = (\alpha'' \cdot \eta)\eta + \underbrace{(\alpha'' \cdot T)}_{0}T + (\alpha'' \cdot U)U$$
$$:= -k_g \eta + k_n U$$

normal curvature $k_n := \alpha'' \cdot U$
geodesic curvature $k_g := -\alpha'' \cdot \eta$

$$\boxed{\begin{array}{l}\alpha \text{ is a geodesic iff } k_g \equiv 0, \\ \text{i.e. } \alpha''(s) \parallel U(\alpha(s))\end{array}}$$

you can "see" this

Example planar curve



$$T' = (k_g)(-\eta)$$

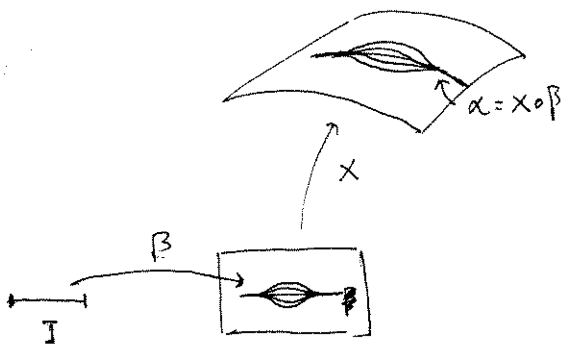
and $T = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ $\quad k_g = \frac{d\theta}{ds}$ = planar curvature.

$$T' = \frac{d\theta}{ds}\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$k_g \equiv 0 \iff \alpha$ is a line

so, k_g is a generalization of planar curvature to surfaces, and geodesics generalize straight lines.

Extrinsic derivation of geodesic condition



$\alpha^t(s) = X(\beta(s) + t\vec{w}(s))$

$\alpha^0(s) = \alpha(s)$ pbal

expand in t: $\vec{w}(s) = \langle u(s), v(s)\rangle$

$$\alpha^t(s) = X(\beta(s)) + t\left[X_u \mid X_v\right]\begin{bmatrix} u(s) \\ v(s) \end{bmatrix} + O(t^2)$$

$$= \alpha(s) + t\,Z(s) + O(t^2)$$

where $Z(s)$ is a vector field along α, $Z(s) \in T_{\alpha(s)}M$

$$\left|\frac{d}{ds}\alpha^t(s)\right| = \sqrt{\alpha^t(s)' \cdot \alpha^t(s)'}$$

$$= \left\{(\alpha' + tZ' + O(t^2)) \cdot (\alpha' + tZ' + O(t^2))\right\}^{1/2}$$

$$= (1 + 2t\,\alpha'\cdot Z' + O(t^2))^{1/2}$$

$$= 1 + t\,\alpha'\cdot Z' + O(t^2) \qquad (\text{Taylor } \sqrt{1+x} = 1 + \tfrac{1}{2}x + O(x^2))$$

$$\Rightarrow L(\alpha^t) = \int_0^L 1 + t\,\alpha'\cdot Z' + O(t^2)\,ds$$

$$= L + t\int_0^L \alpha'(s)\cdot Z'(s)\,ds + O(t^2)$$

$$\Rightarrow \boxed{\begin{aligned} \frac{d}{dt}L(\alpha^t)\Big|_{t=0} &= \int_0^L \alpha'(s)\cdot Z'(s)\,ds \\ &= \alpha'\cdot Z\Big]_0^L - \int_0^L \alpha''(s)\cdot Z(s)\,ds \end{aligned}}$$

this term is zero when $Z(0) = Z(L) = 0$

thus, for fixed endpoint deformations ($\vec{w}(0) = Z(0) = w(L) = Z(L) = 0$),

$$\frac{d}{dt}L(\alpha^t)\Big|_{t=0} = -\int_0^L \alpha''\cdot Z\,ds$$

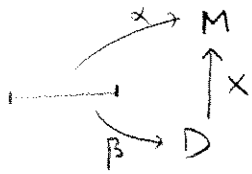
<u>If</u> $k_g = 0$, $\alpha'' \perp T_pM \Rightarrow \alpha''\cdot Z \equiv 0$

$\Rightarrow \frac{d}{dt}L(\alpha^t)\Big|_{t=0} = 0$

<u>If</u> $\frac{d}{dt}L(\alpha^t)\Big|_{t=0} = 0 \ \forall Z$ tangential, then $\alpha'' \perp T_pM \ \forall s$

i.e. $k_g \equiv 0$

Intrinsic eqtns for geodesic



$$\alpha = X \circ \beta$$

$$\alpha' = X_{,i}\,\dot\beta^i$$

$$\alpha'' = X_{,ij}\,\dot\beta^i\dot\beta^j + X_{,i}\,\ddot\beta^i$$

$$= \left(\Gamma^k_{ij} X_{,k} + h_{ij}\vec U\right)\dot\beta^i\dot\beta^j + X_{,k}\,\ddot\beta^k$$

see "Geodesic eqtns" display, p. 230 of text

$$\boxed{\alpha'' /\!/ U \text{ iff}} \quad \boxed{\ddot\beta^k + \Gamma^k_{ij}\dot\beta^i\dot\beta^j = 0 \qquad k=1,2}$$

the Christoffel symbols make the $\vec\beta$ curve on the "map", "bend".

For your reading pleasure:
completely intrinsic derivation of geodesic eqtns: (i.e., only using $[g_{ij}]$)

$$\alpha^t = X\left(\beta + t\vec v(s)\right)$$

$$L(\alpha^t) = \int_0^L \sqrt{\underset{\substack{\uparrow\\ \text{at}\\ \beta(s)+t\vec v(s)}}{g_{ij}}(\dot\beta^i + t\dot v^i)(\dot\beta^j + t\dot v^j)}\,ds \qquad \longleftarrow \text{abbreviate } \tfrac{d}{ds}(\ ) \text{ with } (\dot{\ })$$

$$= \int_0^L \sqrt{\Big(\underset{\substack{\uparrow\\ \text{at }\beta(s)}}{g_{ij}} + \underset{\substack{\uparrow\\ \text{affine}\\ \text{approx}}}{t\,g_{ij,k}v^k} + O(t^2)\Big)(\dot\beta^i + t\dot v^i)(\dot\beta^j + t\dot v^j)}\;ds$$

$$= \int_0^L \sqrt{\underbrace{g_{ij}\dot\beta^i\dot\beta^j}_{\substack{=1\\ (\alpha\text{ pbal})}} + t\Big(\underbrace{g_{ij,k}v^k\dot\beta^i\dot\beta^j + g_{ij}\dot v^i\dot\beta^j + g_{ij}\dot v^j\dot\beta^i}\Big) + O(t^2)}\;ds$$

$$= \int_0^L 1 + \tfrac{1}{2}t\,(\%) + O(t^2)\;ds$$

$$\frac{d}{dt}L(\alpha^t)\Big|_{t=0} = \int_0^L \tfrac{1}{2}\left(g_{ij,k}v^k\dot\beta^i\dot\beta^j + g_{ij}\dot v^i\dot\beta^j + g_{ij}\dot v^j\dot\beta^i\right)ds$$

$$= \int_0^L \tfrac{1}{2}\Big[\Big(g_{ij,k}v^k\dot\beta^i\dot\beta^j - v^i\underbrace{(g_{ij}\dot\beta^j)^{\cdot}}_{\substack{g_{ij,k}\dot\beta^k\dot\beta^j\\ +\,g_{ij}\ddot\beta^j}} - v^j\underbrace{(g_{ij}\dot\beta^i)^{\cdot}}_{\substack{g_{ij,k}\dot\beta^k\dot\beta^i\\ +\,g_{ij}\ddot\beta^i}}\Big]ds \qquad \text{(integ. by parts)}$$

$$= \int_0^L -v^k\Big[g_{kj}\ddot\beta^j + \tfrac{1}{2}\left(-g_{ij,k}\dot\beta^i\dot\beta^j + g_{kj,r}\dot\beta^r\dot\beta^j + g_{ik,r}\dot\beta^r\dot\beta^i\right)\Big]ds$$

iff α is stationary $\forall\, \vec{v}$,

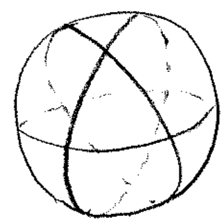
$$g_{kj}\ddot{\beta}^j + \tfrac{1}{2}\left(-g_{rj,k} + g_{kj,r} + g_{jk,r}\right)\dot{\beta}^r\dot{\beta}^j = 0 \qquad k = 1,2$$

$g^{\ell k}$ (%):

$$\boxed{\ddot{\beta}^\ell + \Gamma^\ell_{rj}\dot{\beta}^r\dot{\beta}^j = 0} \qquad \ell = 1,2$$

yipes!
[one should find a better way to do intrinsic Calculus
~ "covariant derivative"]
(next lecture!)

Examples



sphere: great circles satisfy $\alpha'' \,//\, U$

every geodesic is a great circle:

α pbal $\quad \alpha'' // U // \alpha \quad \Rightarrow (\alpha \times \alpha')' = 0$

$\Rightarrow \alpha \times \alpha' = \vec{c} \quad (\& \ \vec{c} \neq 0)$

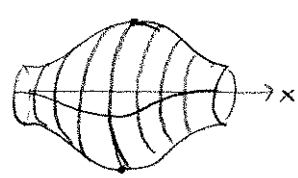
$\Rightarrow (\alpha \cdot \vec{c}) \equiv 0$

$\Rightarrow \alpha$ lies on plane thru origin! ∎

hyperbolic space: ? (good question)

cylinder } wrap up paper
cone }

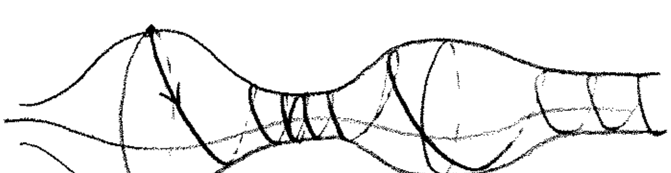
surfaces of revolution



u-coord curves
all meridians are geodesics
max & min parallels.
(not all parallels!)
else?

$$X\begin{pmatrix}u\\v\end{pmatrix} = \begin{bmatrix} x(u) \\ y(u)\cos v \\ y(u)\sin v \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix} \text{ pbal}$$

$$[g_{ij}] = \begin{bmatrix} 1 & 0 \\ 0 & y^2(u) \end{bmatrix} \rightsquigarrow \Gamma^{ij}_k$$



special case of page 230

$$\begin{cases} u'' - (yy_u)(v')^2 = 0 \\ v'' + \left(\frac{y_u}{y}\right) u'v' = 0 \\ (u')^2 + y^2(v')^2 = 1 \end{cases}$$

I'll make a HW problem out of this.

$$\frac{dv}{du} = \frac{\pm c}{y\sqrt{y^2 - c^2}} \quad \text{along geodesics}$$

page 162 Clairaut relation.

larger y $\Rightarrow$ smaller $\frac{dv}{du}$

Physics : A particle constrained to lie on a surface (but with no other external forces), satisfies

$$\alpha''(t) = f(t)U(t) \qquad \text{Newton!}$$

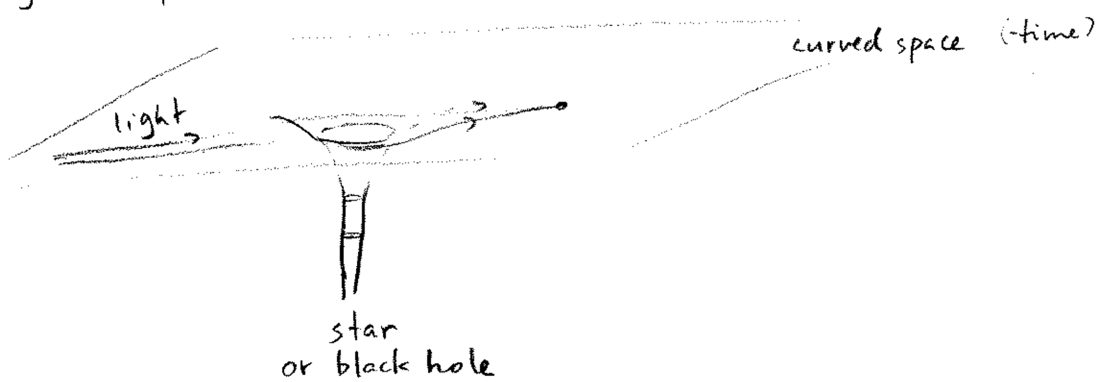
this is geodesic eqtn!

notice if $\|\alpha'(0)\| = 1$

then $\frac{d}{dt}(\alpha' \cdot \alpha') = 2\alpha' \cdot \alpha'' = 0$, so α pbal.

$\alpha' \in T_p(M)$, $\alpha'' \parallel U$

then,
light is a particle. ($E = mc^2$)



how many geodesics are there?

physics intuition : you should be able to start at any point and go in any direction

math proof :

Consider the 1st order system IVP

$$\begin{cases} \dot{\beta}^{\ell} = v^{\ell} & \ell = 1,2 \\ \dot{v}^{k} + \Gamma^{k}_{ij} v^i v^j = 0 & k = 1,2 \\ \vec{\beta}(0) = \vec{q} & \\ \vec{v}(0) = \vec{v}_0 & \text{s.t. } v_0^i v_0^j g_{ij}(q) = 1 \end{cases}$$

$\exists!$ soltn to IVP, and in fact $\alpha := X \circ \beta$ is pbal, hence satisfies geodesic eqtn

check $\frac{d}{dt}\left(g_{ij}\dot{\beta}^i \dot{\beta}^j\right) \equiv 0$ from the system.