Math 4530
Wed 13 April.

Extract geometry from W.E. representation:

$$\Phi(z) = \begin{bmatrix} f(1-g^2) \\ if(1+g^2) \\ 2fg \end{bmatrix} = X_u - iX_v$$

$$X(u,v) = \operatorname{Re} \int_p^z \bar{\Phi}(z)\,dz \qquad (+\vec{c})$$

$St \circ U \circ X = g$ (g for Gauss)

g determines tangent plane

f rotates & dilates it.

Gauss curvature

Note: $\Phi \cdot \bar{\Phi} = (X_u - iX_v)\cdot(X_u + iX_v) = |X_u|^2 + |X_v|^2 = 2\varphi^2$

$$\begin{aligned}
&\| \\
&f(1-g^2)(\bar{f})(1-\bar{g}^2) + if(1+g^2)(-i\bar{f})(1+\bar{g}^2) + 2fg\,2\bar{f}\bar{g} \\
&= |f|^2\left(1+|g|^4 - \cancel{g^2} - \cancel{\bar{g}^2}\right) + |f|^2\left(1+|g|^4 + \cancel{g^2} + \cancel{\bar{g}^2}\right) + 4|f|^2|g|^2 \\
&= 2|f|^2(1+|g|^2)^2
\end{aligned}$$

so conformal scaling factor for $X = |f|(1+|g|^2)$

for $U = |k_1| = \sqrt{-K}$ (Minimal!)

for $St = \dfrac{x^2+y^2+1}{2}$ (reciprocal of that for St^{-1} which you did in old HW)

$= \dfrac{|g|^2+1}{2}$

$$\begin{tikzcd}
M^2 \arrow[r, "U"] & S^2 \arrow[d, "St"] \\
\mathbb{C} \arrow[u, "X"] \arrow[r, "g"'] & \mathbb{C}
\end{tikzcd}$$

Equate scaling:

$$|f|(1+|g|^2)\sqrt{-K}\left(\frac{|g|^2+1}{2}\right) = |g'|$$

* for any analytic map the scaling factor is just the derivative's magnitude – see our approximation discussion.

$$\boxed{K = -\frac{4|g'|^2}{|f|^2(1+|g|^2)^4}}$$

could've used this to actually get K for all minimal surfaces in earlier HW

c.f. uncircled HW 4.8.21 to get this from the Gauss Thm Egregium formula.

example $(f,g) = (1, z)$ Enneper

$$K = \frac{-4\cdot 1}{(1+|z|^2)^4} = \frac{-4}{(1+u^2+v^2)^4} \checkmark$$

Conjugate surfaces

If X arises from the complex pair (f,g)
then there is a family of surfaces

$$X^t \text{ from } (e^{it} f, g)$$

with <u>same</u> Gauss map
<u>same</u> scaling factor for each X^t (hence all surfaces are isometric!)
tangent planes just rotated, not scaled.

e.g. our friend Helcat

Catenoid $(f,g) = \left(-\frac{e^{-z}}{2}, -e^{z}\right)$

we computed this from Maple

from which we could've deduced this from Φ
on HW you check this yields Catenoid

$$X(u,v) = \langle e^u \cos v, e^u \sin v, u \rangle$$

we would check 3rd component!

Helcat family: $(f,g) = \langle e^{it}\left(-\frac{e^{-z}}{2}\right), -e^{z} \rangle$
at $t = \pi/2$ get helicoid (see Maple)

Weierstrass - Enneper II (See text p. 203)

$$\begin{tikzcd} M^2 \arrow[r, "\vec{U}"] & S^2 \arrow[d, "st"] \\ \underset{(u,v)}{\mathbb{C}} \arrow[u, "X"] \arrow[r, "g"] & \underset{(x,y)}{\mathbb{C}} \arrow[ul, bend left=60, "Y"'] \end{tikzcd}$$

new patch $Y := X \circ g^{-1}$ (change of coords) for same surface.

Yields new gauss map $G = id$!

If we write $\tau = x + iy$

$$G(\tau) = \tau$$

$$\Phi^{II}(\tau) = Y_x - iY_y$$

$$= \begin{bmatrix} F(1-\tau^2) \\ iF(1+\tau^2) \\ 2F\tau \end{bmatrix}$$

$$Y(x,y) = Re \int_q^\tau \Phi^{II}(\tau)\, d\tau$$

So really (& locally), each $\mathbb{C}$-analytic fcn corresponds to a minimal surface, & vise verse!

It turns out $F(\tau) = \dfrac{f(g^{-1}(\tau))}{g'(g^{-1}(\tau))} = \dfrac{f(z)}{g'(z)}$

Since chain rule $\Rightarrow$

$$\Phi^{II}(\tau) = Y_x - iY_y$$

$$= X_u u_x + X_v v_x - i(X_u u_y + X_v v_y)$$

$$= (X_u - iX_v)(u_x + iv_x) \qquad (CR)$$

$$= \Phi^{I}(z)\,(g^{-1})'$$

$$= \Phi^{I}(z)\,\frac{1}{g'(z)}$$

i.e.

$$\begin{bmatrix} F(1-\tau^2) \\ iF(1+\tau^2) \\ 2F\tau \end{bmatrix} = \begin{bmatrix} f(1-g^2) \\ if(1+g^2) \\ 2fg \end{bmatrix} \frac{1}{g'(z)} = \begin{bmatrix} f(1-\tau^2) \\ if(1+\tau^2) \\ 2f\tau \end{bmatrix} \frac{1}{g'(z)}$$

$$F = \frac{f}{g'} \quad !$$