

Math 4530
Wed 13 April.

①

Extract geometry from W.E. representations:

$$\Phi(z) = \begin{bmatrix} f(1-g^2) \\ if(1+g^2) \\ 2fg \end{bmatrix} = X_u - iX_v$$

$$X(u,v) = \operatorname{Re} \int_P^z \bar{\Phi}(z) dz \quad (+ \bar{z})$$

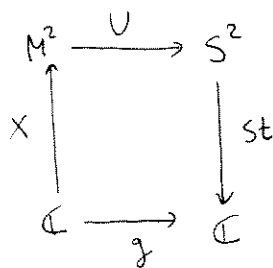
$St \circ U \circ X = g$ (g for Gauss)
g determines tangent plane
f rotates & dilates it.

Gauss curvature

Note: $\Phi \cdot \bar{\Phi} = (X_u - iX_v) \cdot (X_u + iX_v) = |X_u|^2 + |X_v|^2 = 2g^2$

$$\begin{aligned} &= f(1-g^2)(\bar{f})(1-\bar{g}^2) + if(1+g^2)(-i\bar{f})(1+\bar{g}^2) + 2fg \cdot 2\bar{f}\bar{g} \\ &= |f|^2(1+|g|^4 - g^2 - \bar{g}^2) + |f|^2(1+|g|^4 + g^2 + \bar{g}^2) + 4|f|^2|g|^2 \\ &= 2|f|^2(1+|g|^2)^2 \end{aligned}$$

so conformal scaling factor for $X = |f|(1+|g|^2)$
for $U = |k| = \sqrt{-K}$ (Minimal!)
for $St = \frac{x^2+y^2+1}{2}$ (reciprocal of that for St^{-1} , which you did in old HW)
 $= \frac{|g|^2+1}{2}$



Equate scaling:

$$|f|(1+|g|^2)\sqrt{-K} \left(\frac{|g|^2+1}{2} \right) = |g'|$$

$$K = - \frac{4|g'|^2}{|f|^2(1+|g|^2)^4}$$

could've used this to actually get K for all minimal surfaces in earlier HW

c.f. uncircled HW 4.8.21 to get this from the Gauss Theorem Egregium formula.

for any analytic map the scaling factor is just the derivative's magnitude - see our approximation discussion.

example $(f, g) = (1, z)$ Enneper

$$K = \frac{-4 \cdot 1}{(1+|z|^2)^4} = \frac{-4}{(1+u^2+v^2)^4} \quad \checkmark$$

Conjugate surfaces

(2)

If X arises from the complex pair (f, g)

then there is a family of surfaces

$$X^t \text{ from } (e^{it}f, g)$$

with same Gauss map

same scaling factor for each X^t (hence all surfaces are isometric!)

tangent planes just rotated, not scaled.

e.g. our friend Helcat

$$\text{Catenoid } (f, g) = \left(-\frac{e^{-z}}{2}, -e^z\right)$$

↑
we computed this from
Maple

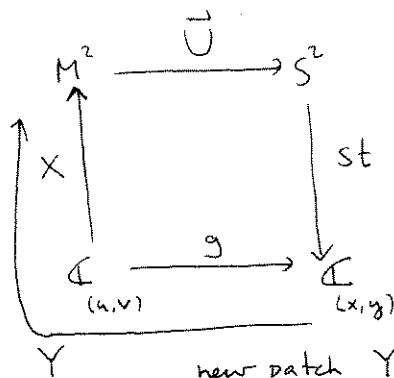
from which we could've deduced this from Φ
on HW you check this yields Catenoid

$$X(u, v) = \langle e^u \cos v, e^u \sin v, u \rangle$$

we could check 3rd component!

$$\text{Helcat family: } (f, g) = \langle e^{it}(-\frac{e^{-z}}{2}), -e^z \rangle$$

at $t = \pi/2$ get helicoid (see Maple)



new patch $Y := X \circ g^{-1}$ (change of coords) for same surface.

Yields new gauss map $G = id!$

If we write $\tau = x + iy$

$$G(\tau) = \tau$$

$$\Phi^{\Pi}(\tau) = Y_x - iY_y$$

$$= \begin{bmatrix} F(1-\tau^2) \\ iF(1+\tau^2) \\ 2F\tau \end{bmatrix}$$

$$Y(x,y) = \operatorname{Re} \int_{\gamma} \Phi^{\Pi}(\tau) d\tau$$

So really (& locally),
each $\mathbb{C}$ -analytic fcn
corresponds to a minimal
surface, & vice versa!

$$\text{It turns out } F(\tau) = \frac{f(g'(\tau))}{g'(g'(\tau))} = \frac{f(z)}{g'(z)}$$

Since chain rule $\Rightarrow$

$$\Phi^{\Pi}(\tau) = Y_x - iY_y$$

$$= X_u u_x + X_v v_x - i(X_u u_y + X_v v_y)$$

$$= (X_u - iX_v)(u_x + iv_x) \quad (CR)$$

$$= \Phi^I(z) (g')'$$

$$= \Phi^I(z) \frac{1}{g'(z)}$$

i.e.

$$\begin{bmatrix} F(1-\tau^2) \\ iF(1+\tau^2) \\ 2F\tau \end{bmatrix} = \begin{bmatrix} f(1-g^2) \\ if(1+g^2) \\ 2fg \end{bmatrix} \frac{1}{g'(z)} = \begin{bmatrix} f(1-\tau^2) \\ if(1+\tau^2) \\ 2f\tau \end{bmatrix} \frac{1}{g'(z)}$$

$$F = \frac{f}{g'}$$