Math 4530 : Minimal surfaces and complex analysis

Monday 4/11

HW for this Friday 4/15

HW1 HW2 HW3 } 4/6 notes

HW4 today's notes

4.8.10
4.8.11
4.8.15
4.8.21
4.8.25

Review

- $g(u+iv) = g(z) = x(u+iv) + iy(u+iv)$

  analytic at z iff

  (1) $\lim\limits_{\substack{h\to 0\\ \in \mathbb{C}}} \dfrac{g(z+h)-g(z)}{h} := g'(z)$ exists $= \begin{cases} g_u & h=\Delta u \text{ real} \\ -ig_v & h = i\Delta v \text{ imag.}\end{cases}$

  (2) g is conformal & orientation preserving i.e.

  $$\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \sqrt{a^2+b^2}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

  $$\boxed{\begin{aligned} x_u &= y_v \\ x_v &= -y_u \end{aligned}} \qquad \boxed{g_v = i g_u}$$

- Conformal maps $F: M^2 \to N^2$; orientation preserving (reversing) maps; conformal (isothermal) coords $X: \mathcal{U} \to M^2$

- If X is a conformal patch of a minimal surface & St is stereographic proj.

  then $\underbrace{St \circ U \circ X}_{:= g}$ is conformal and orientation preserving, hence analytic

Today we seek to reverse these ideas – start with analytic function g and construct a minimal surface.

The first step is to see that knowing a Φ yields an X:

Recall page 7

- X is a minimal isothermal patch iff

$$* \begin{cases} \Phi : X_u - i X_v \\ \quad \text{satisfies} \\ \Phi \cdot \Phi = 0 \quad (\text{isothermal}) \\ \Phi \text{ analytic in each component} \quad (\text{minimal}) \end{cases}$$

<u>Theorem</u> : (Complex analytic function magic)

① If $f(z)$ is analytic, so is $f', f'', f''', \ldots$

② If $f(z)$ is analytic on a simply-connected domain (no holes), then it has an antiderivative

$$F(z) = F(p) + \underbrace{\int_p^z f(z)\,dz}_{}$$

the line integral, for $f(z) = f(u+iv) = x(u,v) + i y(u,v)$

$$:= \int_p^z (x(u,v) + i y(u,v))(du + i\,dv)$$

$$= \int_p^z x\,du - y\,dv \; + i \int_p^z y\,du + x\,dv$$

path independence ?

$-y_u = x_v$ ✓ CR $x_u = y_v$ ✓

Also, since the usual diff rules apply for $\mathbb{C}$-analytic fcns, you can also get the antiderivative symbolically, in many cases.

<u>Theorem</u> : Given Φ satisfying $\begin{cases} \Phi \cdot \Phi = 0 \\ \Phi \text{ analytic} \end{cases}$; $\Phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$

Let $\vec{F}(z)$ be an antiderivative of Φ (in each component). Note F is unique up to additive const.

Then $X(u,v) := Re(F(u+iv))$ is a minimal isothermal patch

$$= Re\left(\int \Phi(z)\,dz\right).$$

<u>pf</u> $X_u = Re(F_u) = Re(\Phi)$ CR! $F_u = F' = \Phi$

$X_v = Re(F_v) = Re(i\Phi)$ $F_v = i F_u = i\Phi$

$= -Im(\Phi)$

So $\Phi = X_u - i X_v$.

So $* \Rightarrow X$ is minimal isothermal ∎

Example
Enneper revisited.

from page 7, (did Friday)

$$X(u,v) = \begin{bmatrix} u - \frac{1}{3}u^3 + uv^2 \\ -v - u^2 v + \frac{1}{3}v^3 \\ u^2 - v^2 \end{bmatrix}$$

$$X_u = \begin{bmatrix} 1-u^2+v^2 \\ -2uv \\ 2u \end{bmatrix}$$

$$X_v = \begin{bmatrix} 2uv \\ -1-u^2+v^2 \\ -2v \end{bmatrix}$$

$$\Phi := X_u - iX_v = \begin{bmatrix} 1 - z^2 \\ i(1+z^2) \\ 2z \end{bmatrix}$$

$$\begin{cases} \Phi \cdot \Phi = 0 \\ \Phi \text{ analytic.} \end{cases}$$

Reverse process.

$$\int \Phi \, dz = \begin{bmatrix} z - \frac{z^3}{3} + c_1 \\ i(z + \frac{z^3}{3}) + c_2 \\ z^2 + c_3 \end{bmatrix}$$

$$= \begin{bmatrix} u+iv - \frac{(u+iv)^3}{3} \\ i(u+iv + \frac{(u+iv)^3}{3}) \\ (u+iv)^2 \end{bmatrix} \quad \text{if } \vec{c} = 0$$

$$X = Re\left(\int \Phi \, dz\right)$$

$$= \begin{bmatrix} u - \frac{u^3}{3} + uv^2 \\ -v + \frac{u^3}{3} - uv^2 \\ u^2 - v^2 \end{bmatrix} \quad \checkmark \text{ Enneper.}$$

We can use the Enneper template to make a good Φ for any analytic fcn $g(z)$!

$$\Phi := X_u - iX_v := \begin{bmatrix} 1 - g^2 \\ i(1+g^2) \\ 2g \end{bmatrix}$$

check $\Phi \cdot \Phi = 0$ ✓
Φ analytic

Furthermore: $St \circ U \circ X = g$!

This must be true, because we checked on MAPLE that for Enneper

$$St \circ U \circ X(z) = z$$

But you can also check this directly, see page 10 of notes for hints.
This is HW 4