HW for Fri 4/8
4.3.5 4.3.6 4.4.3
possibly some more on Monday

Math 4530
Friday April 1.

Sorry - bubbles & soap films Monday. - Next week will be minimal surfaces week.

Today: Ros' proof of Alexandrov's Thm,
that the only (regular) compact surface of const. mean curv is a sphere.

set up: Finish page 4 Wed which gives the energy formulation of constant mean curvature surfaces, corresponding to the Newton's law derivation from a week ago.

In Hw (4.4.3), you give another formulation:

Thm M has constant mean curvature iff $\left.\frac{dA}{dt}\right|_{t=0} = 0$ for all deformation fields $\vec{V}$ for which $\left.\frac{d\,Vol}{dt}\right|_{t=0} = 0$.

(if H = constant this condition is implied, because

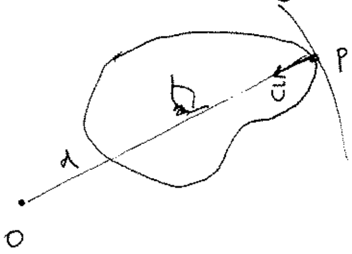
$$\left.\frac{dA}{dt}\right|_{t=0} = \iint_M -2H\vec{U}\cdot\vec{V}\,dA = -2H \underbrace{\iint_M \vec{U}\cdot\vec{V}\,dA}_{\left.\frac{d\,Vol}{dt}\right|_{t=0} = 0}$$

; In HW you prove the converse.)

Deduce: soap bubbles have constant mean curvature (either from Newton, or from above.)

Lemma: If M^2 is compact & regular, with constant mean curvature H, then $H \neq 0$

proof: (as is K const $\Rightarrow$ sphere thm).
If p is in M^2, at max dist d from origin, then $k_i(p) \geq \frac{1}{d} \Rightarrow H \geq \frac{1}{d}$
($\vec{U}$ = inner normal)



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(details March 21)

Alexandrov's Thm will follow from

<u>Ros' Inequality</u> : Let M^2 have positive (possibly varying) mean curvature H (with respect to <u>inner</u> normal)

Then $$\iint_M \frac{1}{H}\, dA \geq 3\ \text{Vol}$$

where Vol is the volume enclosed by M. (Call this region D)
The case of equality only occurs if M is a sphere.
(check it for a sphere!)

Assuming Ros, we get Alexandrov: Let $\vec{U}$ be the inner normal from now on, so that $H>0$

Consider dilations: (by $1+t$)

$$X^t(u,v) = (1+t)X(u,v) = X(u,v) + t\,(X(u,v))$$

$A(t) = (1+t)^2 A$ (because you're scaling by $1+t$)

$\Rightarrow A'(0) = 2A$

Also, compute $A'(0)$ by taking $\vec{V} = \vec{X} = \vec{x}$ in Wed calc:

$$A'(0) = \iint_M -2H\,\vec{V}\cdot\vec{U}\,dA$$

(notice, if $\vec{U} \to -\vec{U}$, $H \to -H$ so formula holds for either choice of $\vec{U}$)

$$= 2H \iint_M \vec{V}\cdot\underbrace{(-\vec{U})}_{\text{ext normal}}\,dA$$

$$= 2H \iint_M \vec{x}\cdot(-\vec{U})\,dA$$

$$= 2H \iiint_D \underbrace{\operatorname{div}\vec{x}}_{3}\,dV = 2H\ \text{Vol}\cdot 3$$

Thus, H const

$\Rightarrow 2A = 2H\cdot 3\ \text{Vol}$

$$\boxed{A = 3H\ \text{Vol}}$$

But Ros $\Rightarrow \text{Vol} \leq \frac{1}{3}\iint_M \frac{1}{H}\,dA = \frac{1}{3H}A$; = iff M is sphere

$\Rightarrow A = 3H\,\text{Vol} \leq \frac{3H}{3H}A$; = iff iff M is sphere

But we have =, so M is a sphere!

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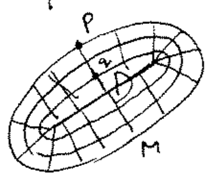
Proof of Ros' inequality

estimate volume of D
by parameterizing from M

innernormal
$\downarrow$

2-d picture



$$Y(u,v,t) = X(u,v) + t\,\vec{U}(u,v)$$

$$0 \le t \le h(p)$$

$$h(p) = \sup\{\, r \mid p \text{ is the nearest point to } q = p + r\vec{U} \,\}$$

This parameterization covers all of D because

let $z \in D$
let $p \in M$ be a nearest point (minimize $f(\vec{x}) = |\vec{x} - \vec{z}|$, $x \in M$)
write $t = |z - p|$
then for $q = p + r\vec{U}$ $\quad 0 \le r < t$
p is the nearest point to q on M (see figure - this is "elementary" geom)

$\Rightarrow h(p) \geq t$
$\Rightarrow z$ is $Y(u,v,t)$ $\blacksquare$

Now estimate Vol, using Y

recall
$$dVol = |\det(Y_u, Y_v, Y_t)|\, dt\,du\,dv$$

$$Vol \underset{(\text{actually} =)}{\le} \iint_M \int_{t=0}^{h(u,v)} |\det(Y_u, Y_v, Y_t)|\, dt\,du\,dv$$

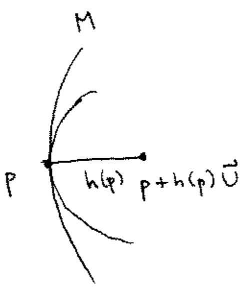
(you're really adding up integrals like this for a partition of patches on M)

$$(Y_u \times Y_v) \cdot Y_t$$
$$= [(X_u + tU_u) \times (X_v + tU_v)] \cdot U$$
$$= \left\{ \begin{array}{l} \left[(X_u \times X_v) + t\left[(-a_1^1 X_u - a_1^2 X_v) \times X_v + X_u \times (-a_2^1 X_u - a_2^2 X_v)\right]\right. \\ \qquad\qquad \left. + t^2 \left[(-a_1^1 X_u - a_1^2 X_v) \times (-a_2^1 X_u - a_2^2 X_v)\right]\right] \end{array} \right\} \cdot \vec{U}$$

$$= (X_u \times X_v)(1 - 2tH + t^2 K) \cdot \vec{U}$$

$$\boxed{\text{so } |dVol| = |1 - 2tH + t^2 K|\, dA_M}$$

$$|\det(Y_u, Y_v, Y_t)| = |1 - 2tH + t^2 K|$$
$$= |(1-k_1 t)(1-k_2 t)|$$



as usual,
$$k_n(p) \leq \frac{1}{h(p)} \leq \frac{1}{t} \quad (\text{since } t \leq h(p))$$
$$\Rightarrow 1 - k_1 t \geq 0$$
$$1 - k_2 t \geq 0$$

$$\Rightarrow dVol = (1-k_1 t)(1-k_2 t)\, dA_M$$
$$= \left(1 - t\left(H + \tfrac{k_1 - k_2}{2}\right)\right)\left(1 - t\left(H - \tfrac{k_1-k_2}{2}\right)\right) dA_M$$
$$= \left[(1-Ht)^2 - \left(\tfrac{k_1-k_2}{2} t\right)^2\right] dA_M \quad (\text{difference of squares!})$$

so,
$$\int_{t=0}^{h(u,v)} (1-k_1 t)(1-k_2 t)\, dt \leq \int_0^{h(p)} (1-Ht)^2\, dt \qquad = \text{ iff } k_1 = k_2$$
$$\leq \int_0^{\frac{1}{H}} (1-Ht)^2\, dt \qquad \left(H \leq \tfrac{1}{h(p)}\right)$$
$$= -\frac{1}{3H}(1-Ht)^3 \Big]_0^{\frac{1}{H}} = \frac{1}{3}H$$

$$\Rightarrow Vol \leq \iint_M \frac{1}{3} H \, dA_M$$

equality iff $k_1 = k_2\ \forall p \in M$, iff surface is totally umbilic, i.e sphere ∎