

Math 4530

Homework due Monday March 18

Professor Smale will substitute for me on Wednesday March 13. I will be away all day. I will be available for our Thursday problem session, **but not until 11:00**, unfortunately. We meet in my office LCB 204 until there are at least 3 people, at which point we move to LCB 215 or another available room.

Here is your homework assignment:

I) Computing curvatures for surfaces: You should develop a skill at doing curvature computations by hand, as in the notes from Monday March 4, pages 3-4. **Also understand the derivations.** The notation introduced there is equivalent to that used in the text, but uses matrices to make the computations more natural. You may wish to compare those notes to pages 89-92 of the text. The maple procedures in the file "surfacecurvatures.mws" follow the matrix notation in the March 4 notes, and allow one to check work, or do hard work, on the computer. I have selected a few problems from Chapter 3; do more if you wish :)

2.4 (just the formulas for K^t and H^t , not what follows after) This is an interesting (and nontrivial) computation which we will use within the next couple of weeks. Get started by noting (proving) that the normal U on the parallel surface agrees with the normal at the corresponding point of the original surface. Also, you can turn dot products of second derivatives of U with U into dot products of first derivatives of U with first derivatives of U if you use the product rule. If you take parallel surfaces to spheres you get concentric spheres. Check as part of this problem that the formulas for mean and Gauss curvature which you have derived for general parallel surfaces give the right answer in the case of spheres.

2.19abc: You are being asked to find the matrices for the first and second fundamental forms, for an arbitrary Monge patch, i.e. the $[g_{ij}]$ and $[h_{ij}]$ matrices, and to then use those matrices to compute K and H . **Do this problem by hand.** (You may check your answers using Maple, but I don't want to see it.)

2.9 Do this by hand, using your answer from 2.19. **Also do it with Maple.** You will want to use the procedures from surfacecurvatures.mws, but there is no need for me to see those procedures again: download, save, and then open that file from Maple, enter its procedures, but do your particular homework computations in a separate file (and window). Make sure to annotate your work when necessary.

Enneper's surface: Have Maple check the book work on pages 93-94, in particular find the matrices of the first and second fundamental forms, as well as the Gauss curvature and the mean curvature. I only want to see the Maple work in this problem, but it would be healthy for you to check the book's work by hand too. Enneper's surface is an interesting minimal surface in that the components of the patch are simple polynomials. (If you've had any complex analysis you might notice that they are actually the real and/or imaginary parts of z^3 and z^2 .) The Enneper patch looks like a potato chip in a neighborhood of the origin, but is not 1-1 on the whole u - v plane, see picture on page 61. Use Maple to draw a nice picture of a piece of Enneper, colored by Gauss curvature (see exercise 7.4).

II) Surfaces of revolution: Professor Smale will discuss this topic on Wednesday, and along the way he will do many of the exercises in section 3 of chapter 3. I would like you to do: **3.5, 3.7, 3.8, 3.10, 3.11.**

III) Surface integrals:

Recall that if $Y(u,v)$ is a parameterization, then the area element dA for surface integration is given by

$$dA = |Y_u \times Y_v| \, du \, dv$$

Read page 4 of Monday March 11 notes, which is an expansion of the discussion on page 90 ; if $X(u,v)$ is a patch for M , and if $U(u,v)=G(u,v)$ is the corresponding unit normal map to the sphere, then the ratio of the surface area element on the sphere divided by the one on M is the absolute value of the Gauss curvature, i.e.

$$|G_u \times G_v| = |K| |X_u \times X_v|$$

(and whether K equals its absolute value or its opposite depends geometrically on whether the point on the surface is elliptic or hyperbolic). Thus, integrals of Gauss curvature over pieces of a surface are actually computing plus or minus spherical areas, under the Gauss map.

(a) Use this reasoning to explain geometrically (**without actually trying to compute the integral directly!**) why for any ellipsoid M of points satisfying

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

it must be that the integral of the Gauss curvature (KdA) over the ellipsoid equals exactly 4π . This is a very special case of the Gauss-Bonnet Theorem which asserts that for any surface diffeomorphic to the sphere (i.e. there is a bijection which is differentiable in each direction), the integral of the Gauss curvature will always be 4π .

(b) Using similar reasoning, explain geometrically why the integral of Gauss curvature over any torus of revolution equals exactly zero (hint: consider the elliptic and hyperbolic regions of the torus separately). This is also a special case of Gauss-Bonnet, which states that you will get zero for the integral of Gauss curvature, on any (perhaps wildly-shaped) torus.

(c) There was a smart classical Greek by the name of Pappus, who proved a theorem which bears his name. For any regular curve α in the x - y plane of length L , p.b.a.l., and for which the y -coordinate is always positive, consider the the center of mass for the curve, namely

$$C := \frac{\int_0^L \alpha(s) \, ds}{L}$$

Let R the distance from C to the x -axis. Pappus' theorem states that the surface area of the surface obtained by rotating α about the x -axis is given by

$$A = 2 \pi R L$$

Prove Pappus' theorem. (This is especially easy if you use Lagrange's identity to compute the surface area element in terms of the $[g_{ij}]$ matrix, for the usual surface of revolution patch.)

(d) What is the surface area of a torus of revolution, in terms of its two radii, r and R , according to Pappus?