Monday September 9

1.5: The Cauchy-Riemann equations, the differential map, chain rules.

Announcements

- hw problems 1.5.8 & 1.5.16 are postponed (as announced on CANVAS)
- If you have questions about HW (new or returned), please visit me in LCB 204 during office hours, or make an appointment

  office hours $\begin{cases} \text{M, W} & 12{:}50 - 1{:}40 \\ \text{T} & 11{:}50 - 12{:}40 \end{cases}$

Warm-up exercise

look over HW ~ hw2 was quite a bit more challenging than hw1.
(most of the course won't be as intense, as far as the 3220 analysis goes)

On Friday we saw that if $f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$ is complex differentiable at $z_0 = x_0 + i\,y_0$,
then u, v have partial derivatives at (x_0, y_0), and these partial derivatives are related by the Cauchy-Riemann equations

$$u_x(x_0,y_0) = v_y(x_0,y_0)$$
$$u_y(x_0,y_0) = -v_x(x_0,y_0).$$

As we saw, the Cauchy-Riemann relations are a consequence of the fact that if the full limit

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} := f'(z_0)$$

exists, then the restricted limits from the real and imaginary directions must also exist and have the same limiting value:

$$f'(z_0) = \lim_{h \to 0,\, h\ real} \frac{f(z_0+h) - f(z_0)}{h} := f_x(z_0) = u_x(x_0,y_0) + i\,v_x(x_0,y_0)$$

$$f'(z_0) = \lim_{i\,h \to 0,\, h\ real} \frac{f(z_0+i\,h) - f(z_0)}{i\,h} := \frac{1}{i} f_y(z_0) = -i\left(u_y(x_0,y_0) + i\,v_y(x_0,y_0)\right).$$

$= v_y(x_0,y_0) - i\,u_y(x_0,y_0)$

There is a converse we used but didn't prove yet, namely

it to show that e^z is analytic & $(e^z)' = e^z$

$e^z = e^{x+iy}$

$e^z = e^x \cos y + i\,e^x \sin y$

Theorem Let

$$F(x,y) = (u(x,y), v(x,y))$$
$$F: A \subseteq \mathbb{R}^2 \to \mathbb{R}^2,\ A \text{ open}$$

Assume

(i) F be differentiable (in multivariable affine approximation sense for real vector-valued functions), at $(x_0, y_0) \in A$. (This is implied if the partial derivatives of u, v exist and are continuous in a neighborhood of (x_0,y_0).)
(ii) The partial derivatives of u, v satisfy the Cauchy-Riemann equations at (x_0, y_0).

Then $f(z) = f(x+iy) := u(x,y) + i\,v(x,y)$ is complex differentiable at $z_0 = x_0 + i\,y_0$ (with $f'(z_0) = f_x(z_0) = -i f_y(z_0)$).

Our general discussion today and of the theorem in particular will use two approximation formulas:

(i) $f'(z_0)$ exists and has value c if and only if we have the affine approximation formula with error estimate:

$L(h) := ch$

$$\boxed{f(z_0 + h) = f(z_0) + c\,h + h\,\varepsilon(h)}$$

where $\varepsilon(h) \to 0$ as $h \to 0$.

check: iff $\lim_{h\to 0} \underbrace{\frac{f(z_0+h) - f(z_0)}{h} - c}_{:= \varepsilon(h)} = 0$

iff $\frac{f(z_0+h) - f(z_0)}{h} - c = \varepsilon(h)$ where $\varepsilon(h) \to 0$ as $h \to 0$

iff $\boxed{f(z_0+h) - f(z_0) - ch = h\,\varepsilon(h)}$ where $\varepsilon(h) \to 0$ as $h \to 0$

And recall

(ii) Def $F(x, y) = (u(x, y), v(x, y))$ is real differentiable at (x_0, y_0) iff we have the affine approximation formula with error estimate for some matrix A:

$$\begin{bmatrix} u(x_0 + h_1, y_0 + h_2) \\ v(x_0 + h_1, y_0 + h_2) \end{bmatrix} = \begin{bmatrix} u(x_0) \\ v(x_0) \end{bmatrix} + A \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + E(h)$$

where $\frac{\| E(h) \|}{\|h\|} \to 0$ as $h \to 0$. If F is differentiable, then the partial derivatives of u, v exist at (x_0, y_0) and the matrix A is the Jacobian matrix

$$A = \begin{bmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{bmatrix}. \quad \underset{=}{CR} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$\vec{F} : \mathbb{R}^n \to \mathbb{R}^m$

$\lim_{\vec{h}\to 0} \frac{\| \vec{F}(\vec{x}_0 + \vec{h}) - \vec{F}(\vec{x}_0) - A\vec{h} \|}{\|\vec{h}\|} = 0$

$\vec{F}$ diff'ble at $\vec{x}_0$ (3220)

<u>The connection between (i) and (ii) in our context:</u>

$h \mapsto f'(z_0)h$

In context (i): Let $f'(z_0) = c = a + i\,b = |c|\,e^{i\theta}$: The linear *differential map* $L(h) = c\,h$ (which transforms tangent vectors based at z_0 into tangent vectors based at $f(z_0)$) is a rotation dilation (as long as $f'(z_0) \neq 0$) where we rotate h by θ and scale it by $|c|$.

$f'(z_0) = |f'(z_0)|\,e^{i\theta}$, $h = \rho e^{i\phi}$

$f'(z_0)h = |f'(z_0)|\,\rho\, e^{i(\theta+\phi)}$

In context (ii) and if the Cauchy-Riemann equations hold, the linear *differential map* $L(\vec{h}) = A\,\vec{h}$ of tangent vectors based at $\vec{x}_0$ to tangent vectors based at $F(\vec{x}_0)$ is a rotation dilation, where the matrix A is the rotation-dilation matrix

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

$L(\vec{h}) = A\vec{h}$

Illustration: Consider $f:\mathbb{C}\to\mathbb{C}$, $F:\mathbb{R}^2\to\mathbb{R}^2$

$$f(z)=z^2=(x+iy)^2, \qquad F(x,y)=(\mathrm{Re}(f),\mathrm{Im}(f))=(x^2-y^2,2xy)$$
$$z_0=1+i, \qquad (x_0,y_0)=(1,1)$$

Compute and illustrate the equivalent affine approximation formulas

$$f(z_0+h)\approx f(z_0)+f'(z_0)h \qquad \vec{F}(\vec{x_0}+\vec{h})\approx\vec{F}(\vec{x_0})+A\vec{h}$$

$z_0=1+i$

$f(z_0)=(1+i)^2=(1+i)(1+i)$
$=2i$

$f'(z)=2z$

$f'(1+i)=2(1+i)$
$=2\sqrt{2}e^{i(\frac{\pi}{4})}$

$h=\rho e^{i\phi}$

approx

$f(z_0+\rho e^{i\phi})\approx f(z_0)+2\sqrt{2}e^{i(\frac{\pi}{4})}\rho e^{i\phi}$
$=f(z_0)+2\sqrt{2}\rho e^{i(\phi+\pi/4)}$

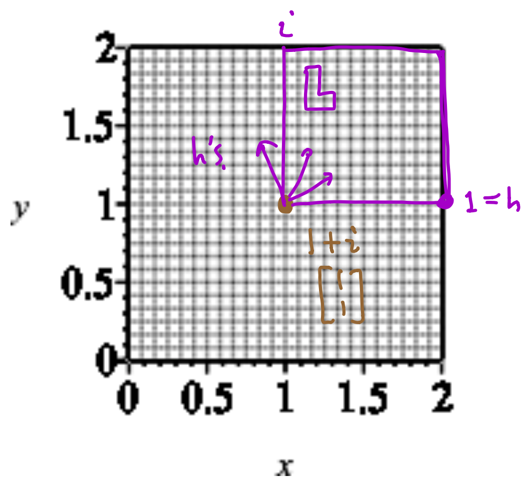
$h\to f'(z_0)h$

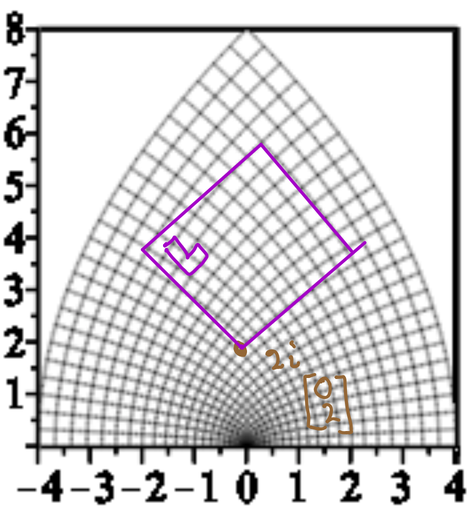
$L(h)$ scaled h by $2\sqrt{2}$
rotated by $\pi/4$

linear

$L(1)=f'(z_0)1=2\sqrt{2}e^{i\pi/4}1$

$L(i)=2\sqrt{2}e^{i\pi/4}e^{i(\frac{\pi}{2})}=2\sqrt{2}e^{i(\frac{3\pi}{4})}$





$F\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}x^2-y^2\\2xy\end{bmatrix}$

$A=\begin{bmatrix}u_x & v_x\\u_y & v_y\end{bmatrix}=\begin{bmatrix}2x & -2y\\2y & 2x\end{bmatrix}$

@ $\begin{bmatrix}1\\1\end{bmatrix}$ $A=\begin{bmatrix}2 & -2\\2 & 2\end{bmatrix}=2\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$

$=2\sqrt{2}\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$

$=2\sqrt{2}\begin{bmatrix}\cos\frac{\pi}{4} & -\sin\frac{\pi}{4}\\\sin\frac{\pi}{4} & \cos\frac{\pi}{4}\end{bmatrix}$

$L(\vec{h})=A\vec{h}$ rotates by $\frac{\pi}{4}$, scales it by $2\sqrt{2}$

Proof of Theorem: Let $F(x, y) = (u(x, y), v(x, y))$ be real differentiable at (x_0, y_0) and so that the Cauchy-Riemann equations hold there. We will show $f(z) = u(x, y) + i\,v(x, y)$ is analytic there and find $f'(z_0)$. We have the affine approximation formula for F with error estimate:

$$\begin{bmatrix} u(x_0 + h_1, y_0 + h_2) \\ v(x_0 + h_1, y_0 + h_2) \end{bmatrix} = \begin{bmatrix} u(x_0) \\ v(x_0) \end{bmatrix} + \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + E(\vec{h})$$

where $\dfrac{\|E(\vec{h})\|}{\|\vec{h}\|} \to 0$ as $\vec{h} \to 0$, and with $a = u_x(x_0, y_0) = v_y(x_0, y_0)$, $b = v_x(x_0, y_0) = -u_y(x_0, y_0)$.

Write $h = h_1 + i\,h_2$ and convert that approximation formula into one for $f(z) = u(x, y) + i\,v(x, y)$:

$$f(z_0 + h) = u(x_0 + h_1, y_0 + h_2) + i\,v(x_0 + h_1, y_0 + h_2)$$

$$= u(x_0, y_0) + a\,h_1 - b\,h_2 + E_1(\vec{h}) + i\cdot\left(v(x_0, y_0) + b\,h_1 + a\,h_2 + E_2(\vec{h})\right)$$

$$= f(z_0) + a\,h_1 - b\,h_2 + i(b\,h_1 + a\,h_2) + E_1(\vec{h}) + i\,E_2(\vec{h})$$

$$f(z_0 + h) = f(z_0) + (a + i\,b)(h_1 + i\,h_2) + h\,\varepsilon(h)$$

which is the affine approximation for f, with $f'(z_0) = a + i\,b$. Note

$$\varepsilon(h) = \frac{E_1(\vec{h}) + i\,E_2(\vec{h})}{h} \to \text{ as } h \to 0$$

because we assumed F was real differentiable at (x_0, y_0).

QED

Chain rules

1) Theorem if f is differentiable at z_0 and g is differentiable at $f(z_0)$ then $g \circ f$ is differentiable at z_0, and

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$$

proof: We use the affine approximation formula for g at $f(z_0)$:

$$g(\underbrace{f(z_0+h)}_{f(z_0)+(f(z_0+h)-f(z_0))}) = g(f(z_0)) + g'(f(z_0))(f(z_0+h) - f(z_0)) + k\,\varepsilon(k) \quad , \quad \varepsilon(k) \to 0 \text{ as } k \to 0$$

for

$$k = f(z_0+h) - f(z_0).$$

rewrite, divide by h:

$$\frac{g(f(z_0+h)) - g(f(z_0))}{h} = g'(f(z_0))\,\frac{f(z_0+h) - f(z_0)}{h} + \frac{k}{h}\varepsilon(k)$$

Take limits as $h \to 0$ and note that the last term $\to 0$ because $\frac{k}{h} \to f'(z_0)$ and $\varepsilon(k) \to 0$, since $k \to 0$ by the continuity of f.

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0).$$

QED

2) (Chain rule for curves) If f is differentiable at z_0 and $\gamma : I \subseteq \mathbb{R} \to \mathbb{C}$ is a parametric curve $\gamma(t) = x(t) + i\,y(t)$ such that $\gamma(t_0) = z_0$ and such that $\gamma'(t_0) = x'(t_0) + i\,y'(t_0)$ exists, then

$$(f \circ \gamma)'(t_0) = f'(\gamma(t_0))\gamma'(t_0)$$

proof We can use the affine approximation formula for f, at $\gamma(t_0)$, and mimic the proof above:

$$f(\gamma(t_0+h)) = f(\gamma(t_0)) + f'(\gamma(t_0))[\gamma(t_0+h) - \gamma(t_0)] + k\,\varepsilon(k) \quad , \quad \varepsilon(k) \to 0$$

because f is diff'ble at $\gamma(t_0)$.

$$\gamma(t_0+h) = \gamma(t_0) + \underbrace{(\gamma(t_0+h) - \gamma(t_0))}_{k}$$

$$\lim_{h \to 0} \quad \underbrace{\frac{f(\gamma(t_0+h)) - f(\gamma(t_0))}{h}}_{(f \circ \gamma)'(t_0)} = f'(\gamma(t_0))\,\frac{\gamma(t_0+h) - \gamma(t_0)}{h} + \frac{k}{h}\varepsilon(k)$$

$$(f \circ \gamma)'(t_0) = f'(\gamma(t_0))\,\gamma'(t_0) + 0$$

<u>Conformal transformations</u> and <u>differentials</u> discussion:

(i) The precise definition of the *tangent space* at $z_0 \in \mathbb{C}$ is the set of all *tangent vectors* there, i.e. tangent vectors to curves passing through z_0:

$$T_{z_0}\mathbb{C} := \left\{ \gamma'(t_0) \,\middle|\, \gamma \text{ is differentiable at } t_0 \text{ and } \gamma(t_0) = z_0 \right\}$$

(ii) If $f(z)$ is a function from $\mathbb{C}$ to $\mathbb{C}$ that arises from a real-differentiable function $F : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2$, then the *differential of f at z_0* is defined by

$$df_{z_0}(\gamma'(t_0)) := (f \circ \gamma)'(t_0).$$

$$df_{z_0} : T_{z_0}\mathbb{C} \rightarrow T_{f(z_0)}\mathbb{C}.$$

(iii) By the chain rule for curves, if $f(z)$ is complex differentiable at z_0, then

$$df_{z_0}(\gamma'(t_0)) := (f \circ \gamma)'(t_0) = f'(z_0)\gamma'(t_0)$$

which is just the rotation-dilation linear part in the affine approximation formula that we were discussing at the beginning of the lecture (or the zero map if $f'(z_0) = 0$). We drew a picture representing the differential map at $z_0 = 1 + i$, for $f(z) = z^2$, near the start of class.

(iv) A function $f : \mathbb{C} \rightarrow \mathbb{C}$ is called *conformal* at z_0 iff its differential transformation preserves angles between tangent vectors. Since rotation-dilations have this property, a function f which is complex differentiable at z_0, and for which $f'(z_0) \neq 0$, is conformal at z_0 .

If we go back to the example from the start of the class, $f(z) = z^2$, the fact that the image curves of the orthogonal coordinate curves are also orthgonal, is a consequence of conformality for $f(z)$, at all $z \neq 0$: