Friday September 6

1.5: complex differentiation; basic properties of analytic functions.

Announcements

- hw2 soltns posted; I'll return assignments Monday

Warm-up exercise

<u>Def</u> Let $f: A \to \mathbb{C}$ where $A \subseteq \mathbb{C}$ is open. Let $z_0 \in A$. We say that f *is (complex) differentiable at* z_0 iff

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} := f'(z_0)$$

exists. Note: an equivalent way to express the limit above is as

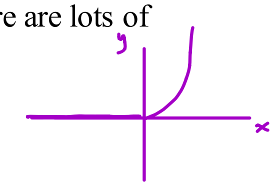
$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

<u>Def</u> Let $f: A \to \mathbb{C}$ where $A \subseteq \mathbb{C}$ is open. If f is complex differentiable for all $z \in A$ then we say that f is *(complex) analytic* or *holomorphic* on A.

<u>Remark</u>: So that you don't get complacent, here is some magic we'll be seeing within a few weeks:

(i) If f is analytic on A as above, then f' is too! And $f'' := (f')'$ is too. And in fact, f has n^{th} order derivatives of every order n on A as soon as its first derivative exists on all of A. Automatically. (Nothing like this was true in general for differentiable functions in regular Calculus! For example there are lots of differentiable functions that are not infinitely differentiable.)

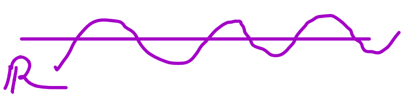
contrast : $x \in \mathbb{R}$ $\quad f(x) = \begin{cases} x^2 & x \geq 0 \\ 0 & x < 0 \end{cases}$ $\quad f'(x) = \begin{cases} 2x & x > 0 \\ 0 & x \leq 0 \end{cases}$



$f''(x) = \begin{cases} 2 & x > 0 \\ 0 & x < 0 \end{cases}$ undefined @ $x = 0$

(ii) If f is analytic on all of $\mathbb{C}$ and if f is also a bounded function, then actually f must be a constant. (This is called Liouville's Theorem.) In fact, if f is analytic on all of $\mathbb{C}$ and if f grows no faster than a polynomial ($|f(z)| \leq C|z|^n$ for $|z| \geq M$ some M), then actually $f(z)$ *is* a polynomial of degree at most n !!

contrast $f(x) = \sin x$ on $\mathbb{R}$



(iii) If f, g are both analytic on and open connected set A and if $\{z_n\}_{n \in N} \subseteq A$ is a sequence of distinct complex numbers, with $\{z_n\} \to z_0 \in A$; and if $f(z_n) = g(z_n)$, $\forall\, n \in \mathbb{N}$, then actually $f(z) = g(z)\ \forall\, z \in A$!!!

$f(z) = z^2$ $\quad f(\frac{1}{n}) = \frac{1}{n^2}$ $\quad n \in \mathbb{N}$ $\quad \{\frac{1}{n}\} \to 0$ in $\mathbb{C}$

$g(z)$ analytic on $\mathbb{C}$ $\quad$ if $g(\frac{1}{n}) = \frac{1}{n^2}$ too then $g(z) = z^2\ \forall z$!!

(iv) If f, g are both analytic on and open connected set A and if the function values and all derivatives of f and g agree at z_0 then actually $f(z) = g(z)$ for all $z \in A$.

Until we get to the magic, let's proceed as we did in Calculus:

<u>Theorem</u> Let f be complex differentiable at $z_0 \in A$, $A \subseteq \mathbb{C}$ open. Then f is continuous at z_0.

pf: If $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.

Then $\lim_{z \to z_0} f(z) - f(z_0) = \lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right)(z - z_0)$

$= f'(z) \cdot 0 = 0$

because limits of prods are prods of limits.

<u>Theorem</u> Let $A \subseteq \mathbb{C}$ open, $f, g : A \to \mathbb{C}$ analytic, $c \in \mathbb{C}$. Then cf, $f+g, fg$ are analytic on A. And the quotient $\frac{f}{g}$ is analytic in A intersect the complement of the zero set for g. Furthermore, for $z \in A$,

(i) $(cf)'(z) = cf'(z)$

(ii) $(f+g)'(z) = f'(z) + g'(z)$

(iii) $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$

(iv) $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$ where $g(z) \neq 0$.

The proofs are just like in Calc 1. We can verify the product rule or the quotient rule, for example:

e.g. (iv) $\left(\frac{f}{g}\right)'(z) = \lim_{h \to 0} \frac{\frac{f(z+h)}{g(z+h)} - \frac{f(z)}{g(z)}}{h}$

$= \lim_{h \to 0} \frac{1}{h} \left[\frac{f(z+h)g(z) - f(z)g(z+h)}{g(z+h)g(z)} \right]$

$= \lim_{h \to 0} \frac{1}{h} \frac{(f(z+h) - f(z))g(z) - f(z)(g(z+h) - g(z))}{g(z+h)g(z)}$

$= \lim_{h \to 0} \frac{g(z)}{g(z+h)g(z)} \frac{f(z+h) - f(z)}{h} - \lim_{h \to 0} \frac{f(z)}{g(z+h)g(z)} \left(\frac{g(z+h) - g(z)}{h} \right)$

$= \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$

note we used limit theorems & continuity of diff'ble fcns

Some more computations that go just like in Calculus:

(i) if $f(z)$ is the constant function, its derivative is zero.

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{z \to z_0} \frac{0}{z-z_0} = 0$$

(ii) if $f(z) = z^n$, $n \in \mathbb{N}$, then $f'(z) = n z^{n-1}$

$$f'(z) = \lim_{h \to 0} \frac{(z+h)^n - z^n}{h} = \lim_{h \to 0} \frac{\cancel{z^n} + n z^{n-1} h + \cdots \binom{n}{k} z^{n-k} h^k + \cdots + h^n - \cancel{z^n}}{h}$$

$$= \lim_{h \to 0} n z^{n-1} + \underbrace{\lim_{h \to 0} \left(\text{sum of terms with a factor of at least } h\right)}_{0}$$

$$= n z^{n-1}$$

(iii) if $f(z) = z^n$, $n \in \mathbb{Z}$, then $f'(z) = n z^{n-1}$

(iv) every polynomial in z is analytic on $\mathbb{C}$, with the expected formula for its derivative.

(iv) if $f(z) = \dfrac{p(z)}{q(z)}$ is a rational function, i.e. a quotient of two polynomials, then $f(z)$ is analytic on the complement of the zero set for q.

just the same!

The chain rule is also true - we'll probably postpone the precise proof until Monday along with a discussion of the inverse function theorem, so that we have time for the rest of today's notes today. (The proof proceeds just like the precise proof for the one real variables chain rule that you discussed in 3210). In any case, if f is differentiable at z_0 and g is differentiable at $f(z_0)$ then $g \circ f$ is differentiable at z_0, and

$$(g \circ f)'(z_0) = g'(f(z_0)) f'(z_0)$$

Now watch out.

Example: Write $z = x + iy$, $y \in \mathbb{R}$. Then $f(z) = \mathrm{Re}(z) = x$ is NOT complex differentiable at any point of $\mathbb{C}$!

$$f'(z) \overset{?}{=} \lim_{\substack{h_1 + i h_2 \to 0 \\ \uparrow \\ h}} \frac{\mathrm{Re}(z+h) - \mathrm{Re}(z)}{h_1 + i h_2}$$

$z = x + iy$
$z + h = x + iy + h_1 + i h_2$
$= (x + h_1) + i(y + h_2)$

$$= \lim_{h_1 + i h_2 \to 0} \frac{\cancel{x} + h_1 - \cancel{x}}{h_1 + i h_2}$$

If full limit exists, it agrees with any limit where I approach $h = 0$ in some restricted way

- $h = h_1 \in \mathbb{R}$: then get $\lim_{h_1 \to 0} \frac{h_1}{h_1} = 1.$
- $(h_1 = 0)$ $h = i h_2$, $h_2 \in \mathbb{R}$ then get $\lim_{i h_2 \to 0} \frac{0}{i h_2} = 0$

So full limit DNE

In fact, being complex differentiable is very rare, relatively speaking, even if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are nice differentiable functions of x and y:

Theorem Let $A \subseteq \mathbb{C}$ open, $f: A \to \mathbb{C}$, $z_0 \in A$. Write $f(z) = f(x + iy) = u(x, y) + i\, v(x, y)$, where $u(x, y) = \mathrm{Re}(f(x + iy))$, $v(x, y) = \mathrm{Im}(f(x + iy))$. Then if f is complex differentiable at $z_0 = x_0 + i y_0$ the following partial derivative equalities - known as the *Cauchy-Riemann equations* - must hold there:

$$u_x(x_0, y_0) = v_y(x_0, y_0)$$
$$u_y(x_0, y_0) = -v_x(x_0, y_0).$$

these eqtns failed in example

$$u_x(x_0, y_0) = \left.\frac{\partial}{\partial x} u(x,y)\right|_{(x,y) = (x_0, y_0)}$$

assuming $f'(z_0)$ exists.

$$f'(z_0) = \lim_{h_1 + i h_2 \to 0} \frac{f(x_0 + i y_0 + h_1 + i h_2) - f(x_0 + i y_0)}{h_1 + i h_2}$$

$f = u + iv$

$$= \lim_{h_1 + i h_2 \to 0} \frac{u(x_0 + h_1, y_0 + h_2) + i\, v(x_0 + h_1, y_0 + h_2) - u(x_0, y_0) - i\, v(x_0, y_0)}{h_1 + i h_2}$$

If $h = h_1 \in \mathbb{R}$.

$$\lim_{h_1 \to 0} \frac{u(x_0 + h_1, y_0) - u(x_0, y_0) + i\left(v(x_0 + h_1, y_0) - v(x_0, y_0)\right)}{h}$$

$f = u + iv$

$$\Rightarrow f'(z_0) = u_x(x_0, y_0) + i\, v_x(x_0, y_0) \quad := f_x(z_0)$$

If $h = i h_2$, $h_2 \in \mathbb{R}$

$$\lim_{h_2 \to 0} \frac{u(x_0, y_0 + h_2) - u(x_0, y_0) + i\left(v(x_0, y_0 + h_2) - v(x_0, y_0)\right)}{i h_2}$$

$$\Rightarrow f'(z_0) = \frac{1}{i} u_y(x_0, y_0) + v_y(x_0, y_0) \quad := \frac{1}{i} f_y(z_0)$$

compare real parts: $u_x(x_0, y_0) = v_y(x_0, y_0)$
compare imag parts: $v_x(x_0, y_0) = -u_y(x_0, y_0)$

As we checked, The Cauchy-Riemann equations are a consequence of complex differentiability. Conversely, if $u(x, y)$ and $v(x, y)$ satisfy the Cauchy Riemann equations, that's almost sufficient to imply complex differentiability. The precise conditions require a bit more, and are connected to your discussions of real variables differentiability from Math 3220:

(proof Monday.)

Theorem Let $A \subseteq \mathbb{C}$ open, $f: A \to \mathbb{C}$, $z_0 \in A$. Write $f(z) = f(x + iy) = u(x, y) + i\,v(x, y)$, where $u(x, y) = \mathrm{Re}(f(x + iy)$, $v(x, y) = \mathrm{Im}(f(x + iy)$. Consider the associated $F(x, y)$ with image in $\mathbb{R}^2$, given by

$$F(x, y) = (u(x, y), v(x, y)),$$

for $x + iy \in A$.

Then $f(z)$ if complex differentiable at z_0 if and only if the Cauchy-Riemann equations hold for the partial derivatives of u and v, at (x_0, y_0) AND $F(x, y)$ is real differentiable at (x_0, y_0) in the multivariable sense. In particular, if the partial derivative functions u_x, u_y, v_x, v_y are continuous in a neighborhood of (x_0, y_0), and if the Cauchy Riemann conditions hold there, then the associated f is complex differentiable at z_0.

Our proof will make use of the facts about differentiability (we'll verify the first one, and recall the second one from Math 3220).

(i) $f'(z_0)$ exists and has value c if and only if we have the affine approximation formula with error estimate:

$$f(z_0 + h) = f(z_0) + c\,h + h\,\varepsilon(h)$$

where $\varepsilon(h) \to 0$ as $h \to 0$.

(ii) $F(x, y) = (u(x, y), v(x, y))$ is real differentiable at (x_0, y_0) and has Jacobian matrix A if and only if we have the affine approximation formula with error estimate some matrix A:

$$\begin{bmatrix} u(x_0 + h_1, y_0 + h_2) \\ v(x_0 + h_1, y_0 + h_2) \end{bmatrix} = \begin{bmatrix} u(x_0) \\ v(x_0) \end{bmatrix} + A \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + E(h)$$

where $\dfrac{\|E(h)\|}{\|h\|} \to 0$ as $h \to 0$. A sufficient condition for real differentiability is that all partial derivatives of u, v exist and are continuous, in a neighborhood of (x_0, y_0).

Example: Show that $f(z) = e^z$ is analytic on $\mathbb{C}$ and find $f'(z)$ by checking the Cauchy Riemann equations and verifying that the partial derivatives of $u(x,y) = \mathrm{Re}(f(z))$, $v(x,y) = \mathrm{Im}(f(z))$ are continuous. (For fun, you could try to use the limit definition of derivative to compute the derivative of e^z directly, but it's more work than you might expect.)

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = e^z$$

direct method not so easy

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y),$$

$$u(x,y) = e^x \cos y$$

$$v(x,y) = e^x \sin y$$

$u_x = v_y$? $\quad u_x = e^x \cos y$, $v_y = e^x \cos y$ ✓

$u_y = -v_x$ $\quad u_y = e^x(-\sin y)$, $v_x = e^x \sin y$ ✓

cont.

verified that all partials are cont. & that they satisfy Cauchy-Riemann eqtns

$$f'(z) = f_x = u_x + i v_x$$

$$= e^x \cos y + i e^x \sin y$$

$$= e^z \ !$$

Example Show that $f(z) = \sin(z)$ is analytic on $\mathbb{C}$ and find its derivative.