Math 4200-001
Week 3concepts and homework
1.5
Due Wednesday September 11 at start of class.

1.5 1ad, 3b, 5c, 6b (in 5c and 6b describe the differential map as a rotation-dilation); 8, 9, 10, 11, 13 (then look at 14 but don't do 14), 16, 18abc, 19.

w3.1 For 6b above, sketch a domain $\mathbb{C}$ and a target $\mathbb{C}$. To illustrate the differential map, add a unit "L-square" to the domain picture, with it's lower left corner at the point i, and sketch its image under the differential map, as a rotated dilated square in the tangent space of $f(i)$.

Math 4200-001
Week 3: Finish section 1.4 and begin section 1.5 on complex derivatives.

Wednesday September 4
Part 2 of section 1.4: Add functions to the mix of sets, sequences, in review of 3220 material we'll be using in 4200.

Announcements
- HW for next week is last page of notes — it's from §1.5
- I only grade some of the HW problems (see public page or CANVAS)
- all solutions are in "files" @ CANVAS

Warm-up exercise

The functions we'll be using Math 4200 will almost always be one of the following three types:

i) For $A \subseteq \mathbb{C}$, $f: A \to \mathbb{C}$ complex functions (or the associated $F: A \to \mathbb{R}^2$, with $A \subseteq \mathbb{R}^2$).

ii) For $I \subseteq \mathbb{R}$, $\gamma: I \to \mathbb{C}$ curves (or the associated $\Gamma: I \to \mathbb{R}^2$).

iii) For $A \subseteq \mathbb{C}$(or $A \subseteq \mathbb{R}^2$), $u: A \to \mathbb{R}$ real-valued functions.

Since limits and continuity are easily discussed for $F: A \to \mathbb{R}^m$ with $A \subseteq \mathbb{R}^n$ we will temporarily adopt that generality, and use the 3220 notation: For $x_0 \in \mathbb{R}^n$,

$$B_\varepsilon(x_0) := \{x \in \mathbb{R}^n \mid \|x - x_0\| < \varepsilon\}$$

instead of the disk notation $D(z_0; \varepsilon)$ which is special to $\mathbb{C}$.

Def Let $A \subseteq \mathbb{R}^n$ and $F: A \to \mathbb{R}^m$.

For $B \subseteq A$, $F(B) := \{y \in \mathbb{R}^m \mid y = F(x) \text{ for some } x \in B\}$ "image of B"

For $C \subseteq \mathbb{R}^n$, $F^{-1}(C) := \{x \in A \mid F(x) \in C\}$ "inverse image of C"

Def Let $A \subseteq \mathbb{R}^n$ and $F: A \to \mathbb{R}^m$, and $x_0 \in A$, $L \in \mathbb{R}^m$. Then $\lim_{x \to x_0} F(x) = L$ iff

- $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $0 < \|x - x_0\| < \delta$ and $x \in A \Rightarrow \|f(x) - L\| < \varepsilon$
- " " " $x \in \left(B_\delta(x_0) \setminus \{x_0\}\right) \cap A \Rightarrow f(x) \in B_\varepsilon(L)$

Example: In section 1.5 we will consider open sets $A \subseteq \mathbb{C}$, $f: A \to \mathbb{C}$, and $z_0 \in A$. We will say that f is (complex) differentiable at z_0 if and only if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} := f'(z_0)$$

exists. We will say that f is analytic on A if it is differentiable for all $z \in A$. This looks like Calculus 1, and the usual differentiation rules work. So, you can do the first few problems of next week's homework already. But it turns out that there are also a lot of surprises in store, as we'll start seeing on Friday and Monday as we cover section 1.5 in depth.

Def Let $A \subseteq \mathbb{R}^n$ and $F: A \to \mathbb{R}^m$, and $x_0 \in A$. Then F *is continuous at* x_0 iff

$$\lim_{\substack{x \to x_0 \\ (x \in A)}} F(x) = F(x_0)$$

$\left(\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } x \in B_\delta(x_0) \cap A \Rightarrow F(x) \in B_\varepsilon(F(x_0))\right)$

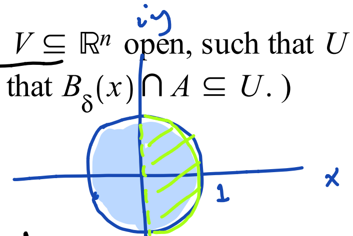
Remark: Showing $\lim_{x \to x_0} F(x) = L$ is logically equivalent to showing $\lim_{x \to x_0} \|F(x) - L\| = 0$, and checking the second statement is often how we'll verify the first one.

means $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $0 < \|x - x_0\| < \delta$ and $x \in A \Rightarrow \big| \|F(x) - L\| - 0 \big| < \varepsilon$, i.e. $\|F(x) - L\| < \varepsilon$

- Def Let $A \subseteq \mathbb{R}^n$ and $F : A \to \mathbb{R}^m$. Then F *is continuous on* A iff F is continuous at each $x_0 \in A$.

Def Let $A \subseteq \mathbb{R}^n$, $U \subseteq A$. Then U *is relatively open (in A)* iff $\exists\ V \subseteq \mathbb{R}^n$ open, such that $U = V \cap A$.
(Note, U is relatively open in A iff $\forall\ x \in U\ \exists\ \delta > 0$ such that $B_\delta(x) \cap A \subseteq U$.)

$A \subset \mathbb{C}$, $A = \overline{D(0;1)}$
$\mathcal{U} = \{z \in A \text{ s.t. } \operatorname{Re} z > 0\}$ is relatively open, even though it includes a bdry semicircle.



Let $A \subseteq \mathbb{R}^n$, $W \subseteq A$. Then W *is relatively closed (in A)* iff $\exists\ V \subseteq \mathbb{R}^n$ closed, such that $W = V \cap A$.

Note
$\mathcal{U} = A \cap \{z \text{ s.t. } \operatorname{Re} z > 0\}$
open

Theorem Let $A \subseteq \mathbb{R}^n$, $F : A \to \mathbb{R}^m$, $x_0 \in A$. The following are equivalent:

(i) F is continuous at x_0 ($\varepsilon - \delta$ definition)
(ii) F is sequentially continuous at x_0, i.e. $\forall\ \{x_n\} \subseteq A$ such that $\{x_n\} \to x_0$, then also $\{F(x_n)\} \to F(x_0)$.

(You essentially proved this theorem in your homework problem 1.4.18., so we can skip it here.)

<u>Theorem</u> Let $A \subseteq \mathbb{R}^n$, $F : A \to \mathbb{R}^m$. The following are equivalent:

(i) F is continuous on A

(ii) For all $O \subseteq \mathbb{R}^m$ open, $F^{-1}(O)$ is (relatively) open in A.

This is an important and useful characterization of contin[u]ous functions, so we'll recall or construct a proof in class.

(i) $\Rightarrow$ (ii) : Let $\mathcal{O} \subset \mathbb{R}^m$. Need to show $F^{-1}(\mathcal{O})$ is open in A.

Let $x \in F^{-1}(\mathcal{O})$. It suffices to show that $\exists \delta > 0$ s.t. $B_\delta(x) \cap A \subset F^{-1}(\mathcal{O})$.

(Because then let $\mathcal{U} = \bigcup_{x \in F^{-1}(\mathcal{O})} B_{\delta_x}(x)$ is open. It contains $F^{-1}(\mathcal{O})$ and $\mathcal{U} \cap A \subset F^{-1}(\mathcal{O})$. So $\mathcal{U} \cap A = F^{-1}(\mathcal{O})$.)

F is continuous at x and $F(x) \in \mathcal{O}$

$\Rightarrow \exists \varepsilon > 0$ s.t. $B_\varepsilon(F(x)) \subset \mathcal{O}$

So $\exists \delta > 0$ s.t. $z \in B_\delta(x) \cap A$

$\Rightarrow F(z) \in B_\varepsilon(F(x)) \subset \mathcal{O}$

$\Rightarrow z \in F^{-1}(\mathcal{O})$.

$B_\delta(x) \cap A \subset F^{-1}(\mathcal{O})$

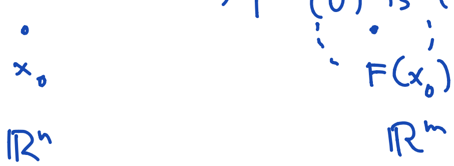
(ii) $\Rightarrow$ (i). Let $x_0 \in A$.

to show F is cont at x_0

Let $\varepsilon > 0$. Let $\mathcal{O} = B_\varepsilon(F(x_0))$

$\Rightarrow F^{-1}(\mathcal{O})$ is (relatively) open in A, i.e. $x_0 \in F^{-1}(\mathcal{O}) \subset \mathcal{U}_{open} \subset \mathbb{R}^n$

$F^{-1}(\mathcal{O}) = \mathcal{U} \cap A$.



$\Rightarrow \exists \delta > 0$

s.t. $B_\delta(x_0) \subset \mathcal{U}$

$B_\delta(x_0) \cap A \subset F^{-1}(\mathcal{O})$

i.e. $x \in A \cap B_\delta(x_0)$

$\Rightarrow F(x) \in \mathcal{O} = B_\varepsilon(F(x_0))$

∎

Theorem $A \subseteq \mathbb{R}^n$, $f, g : A \to \mathbb{R}$; or $F, G : A \to \mathbb{R}^2$ with associated complex functions $f, g : A \to \mathbb{C}$. Then the following limit theorems hold. f, g continous at $x_0 \in A$ implies

(i) $f + g$ is continous at x_0

(ii) fg is continous at x_0

(iii) $\frac{f}{g}$ is continous at x_0 if $g(x_0) \neq 0$.

proof: Since continuity is equivalent to sequential continuity it suffices to consider all $\{x_n\} \subseteq A$ with $\{x_n\} \to x_0$, and assuming $\{f(x_n)\} \to f(x_0)$, $\{g(x_n)\} \to g(x_0)$ to then show that

$$\{f(x_n) + g(x_n)\} \to f(x_0) + g(x_0)$$

$$\{f(x_n)g(x_n)\} \to f(x_0)g(x_0)$$

$$\left\{\frac{f(x_n)}{g(x_n)}\right\} \to \frac{f(x_0)}{g(x_0)}.$$

But this follows from the analogous Theorem for sums, products and quotiens of sequences, which we talked about last Friday. Of course, one could also construct $\varepsilon - \delta$ proofs without using sequential continuity.

Corollary If $p(z)$ is a polynomial function on $\mathbb{C}$, it is continuous on all of $\mathbb{C}$. If $r(z) = \frac{p(z)}{q(z)}$ is a rational function, then it is continuous on all of $\mathbb{C}$, except at the roots of the denominator polynomial $q(z)$.

<u>Remark</u> So that I could appeal to your homework problem, I stated the continuity theorem first. But the same method of proof also implies the analogous limit theorem, even if the functions aren't defined at x_0.

<u>Theorem</u> $A \subseteq \mathbb{R}^n$, $f, g : A \to \mathbb{R}$; or $F, G : A \to \mathbb{R}^2$ with associated complex functions $f, g : A \to \mathbb{C}$. Then if

$$\lim_{x \to x_0} f(x) = L \quad \text{and} \quad \lim_{x \to x_0} g(x) = M,$$

it follows that

(i) $\lim_{x \to x_0} f(x) + g(x) = L + M$

(ii) $\lim_{x \to x_0} f(x)g(x) = LM$

(iii) $\lim_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ provided $M \neq 0$.

Continuous functions and compactness:

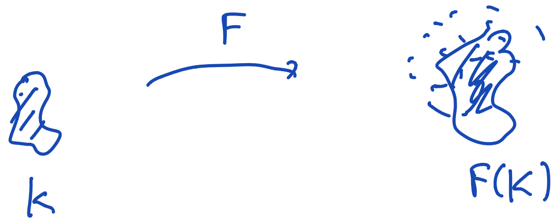
(1) Theorem Let $K \subseteq \mathbb{R}^n$ compact, $F: K \to \mathbb{R}^n$ continuous. Then $F(K) \subseteq \mathbb{R}^m$ is compact.

proof:

Let $\bigcup_\alpha \mathcal{U}_\alpha$ be an open cover of $F(K)$.

then $F^{-1}\left(\bigcup_\alpha \mathcal{U}_\alpha\right) = \bigcup_\alpha F^{-1}(\mathcal{U}_\alpha)$

is an open cover of K.



So it has finite subcover

$$K = F^{-1}(\mathcal{U}_{\alpha_1}) \cup F^{-1}(\mathcal{U}_{\alpha_2}) \cup \cdots F^{-1}(\mathcal{U}_{\alpha_N})$$

$$\Rightarrow F(K) = \mathcal{U}_{\alpha_1} \cup \cdots \cup \mathcal{U}_{\alpha_N}.$$

(2) Corollary (Extreme value Theorem) Let $K \subseteq \mathbb{R}^n$ compact, $F: K \to \mathbb{R}$ continuous. Then F attains its infimum m and supremum M. In other words,

$$\exists\, x_1 \in K \text{ such that } F(x_1) = inf_{x \in K} F(x).$$
$$\exists\, x_2 \in K \text{ such that } F(x_2) = sup_{x \in K} F(x).$$

proof: By Theorem 1, $F(K)$ is compact. Because compact sets in $\mathbb{R}^n$ are bounded the infimum and supremum are finite. Because compact sets are closed, and closed sets contain all their limit points, we know that $F(K)$ contains all its limit points. Thus $F(K)$ contains its infimum and extremum. QED.

We'll start §1.5 on Friday, and discuss the remaining thms in today's notes (and §1.4) as we need them going forward. You've seen them in 3220 also, probably.

Def Let $A \subseteq \mathbb{R}^n$, $F : A \to \mathbb{R}^m$. Then F *is uniformly continuous* iff

(3) Theorem Let $K \subseteq \mathbb{R}^n$, K compact. Let $F : K \to \mathbb{R}^m$ be continous. Then F is uniformly continuous.

proof (Various approaches work, after some effort and depending on the characterization of compact sets you prefer to use. I prefer to use sequential compactness, but the text has an open cover proof.)

Connectivity and functions

The following definition is equivalent to the one we wrote down last week:

A set $A \subseteq \mathbb{R}^n$ is *"not connected"* (or *"has a disconnection"*) iff $\exists\ U, V$ relatively open subsets of A such that

$$A = U \cup V$$

$$U \neq \varnothing,\ V \neq \varnothing$$

$$U \cap V = \varnothing.$$

A set $A \subseteq \mathbb{R}^n$ is *"connected"* if it has no disconnection.

(1) Theorem Let $A \subseteq \mathbb{R}^n$ be connected, $F : A \to \mathbb{R}^m$ continuous. Then $F(A)$ is connected.
proof:

Def Let $A \subseteq \mathbb{R}^n$. A is *path connected* iff $\forall\ x, y \in A,\ \exists\ \gamma : [a, b] \to A$ continuous (a "path") such that $\gamma(a) = x, \gamma(b) = y$.

(2) Theorem If A is path connected, then A is connected.

proof: (sketch) Assume $\{U, V\}$ is a disconnection of A. Pick $x \in A$, $y \in B$ and let $\gamma : [a, b] \to A$ be a (continuous) path connecting x to y. Let

$$c := sup\{t \geq a \mid \gamma([a, t]) \subseteq A\}$$

We can show that $c > a, c < b$ and that $\gamma(c)$ U nor V, using the facts that U, V are both relatively open: Since U is open and $\gamma(a) \in U$, there is a $\delta > 0$ so that $\gamma([a, a + \delta]) \subseteq U$, because γ is continuous. So $c \geq a + \delta$. Similarly, $c < b$. Then, if $\gamma(c) \in U$ the fact that U is open means there is a $\delta>0$ so that $\gamma([c - \delta, c + \delta]) \subseteq U$ which violates the definition of c. So $\gamma(c) \notin U$. Similarly, $\gamma(c) \notin V$.This is a contradiction to the assumption that $\{U, V\}$ covers A. Thus, no disconnection of A exists.

(3) Theorem If A is open and connected, then A is path connected. (Thus, for open sets, being path connected is the same as being connected.)

proof (sketch). Pick any $x_0 \in A$. Let $U = \{y \in A$ such that there is a path from x_0 to $y\}$. We can show that U is open, and that its complement $V := A \setminus U$ which is automatically closed, is also open, unless it is empty. Thus V is empty, since A is connected.