Math 4200-001
Week 3: Finish section 1.4 and being section 1.5 on complex derivatives.

Wednesday September 4
Part 2 of section 1.4: Add functions to the mix of sets, sequences, in review of 3220 material we'll be using in 4200.

Announcements

Warm-up exercise

The functions we'll be using Math 4200 will almost be one of the following three types:

i) For $A \subseteq \mathbb{C}$, $f: A \rightarrow \mathbb{C}$ complex functions (or the associated $F: A \rightarrow \mathbb{R}^2$, with $A \subseteq \mathbb{R}^2$).

ii) For $I \subseteq \mathbb{R}$, $\gamma: I \rightarrow \mathbb{C}$ curves (or the associated $\Gamma: I \rightarrow \mathbb{R}^2$).

iii) For $A \subseteq \mathbb{C}$(or $A \subseteq \mathbb{R}^2$), $u: A \rightarrow \mathbb{R}$ real-valued functions.

Since limits and continuity are easily discussed for $F: A \rightarrow \mathbb{R}^m$ with $A \subseteq \mathbb{R}^n$ we will temporarily adopt that generality, and use the 3220 notation: For $x_0 \in \mathbb{R}^n$,

$$B_\varepsilon(x_0) := \{x \in \mathbb{R}^n \mid \|x - x_0\| < \varepsilon\}$$

instead of the disk notation $\mathrm{D}(z_0; \varepsilon)$ which is special to $\mathbb{C}$.

<u>Def</u> Let $A \subseteq \mathbb{R}^n$ and $F: A \rightarrow \mathbb{R}^m$.

For $B \subseteq A$, $F(B) := \{y \in \mathbb{R}^m \mid y = F(x) \text{ for some } x \in B\}$ "image of B"

For $C \subseteq \mathbb{R}^n$, $F^{-1}(C) := \{x \in A \mid f(x) \in C\}$ "inverse image of C"

<u>Def</u> Let $A \subseteq \mathbb{R}^n$ and $F: A \rightarrow \mathbb{R}^m$, and $x_0 \in A$, $L \in \mathbb{R}^m$. Then $\lim_{x \rightarrow x_0} F(x) = L$ iff

<u>Def</u> Let $A \subseteq \mathbb{R}^n$ and $F: A \rightarrow \mathbb{R}^m$, and $x_0 \in A$. Then F *is continuous at* x_0 iff

<u>Remark</u>: Showing $\lim_{x \rightarrow x_0} F(x) = L$ is logically equivalent to showing $\lim_{x \rightarrow x_0} \|F(x) - L\| = 0$, and checking the second statement is often how we'll verify the first one.

Def Let $A \subseteq \mathbb{R}^n$ and $F : A \to \mathbb{R}^m$. Then F *is continuous on A* iff F is continuous at each $x_0 \in A$.

Def Let $A \subseteq \mathbb{R}^n$, $U \subseteq A$. Then U *is relatively open (in A)* iff $\exists\ V \subseteq \mathbb{R}^n$ open, such that $U = V \cap A$.
(Note, U is relatively open in A iff $\forall\ x \in U\ \exists\ \delta > 0$ such that $B_\delta(x) \cap A \subseteq U$.)

Let $A \subseteq \mathbb{R}^n$, $W \subseteq A$. Then W *is relatively closed (in A)* iff $\exists\ V \subseteq \mathbb{R}^n$ closed, such that $W = V \cap A$.

Theorem Let $A \subseteq \mathbb{R}^n$, $F : A \to \mathbb{R}^m$, $x_0 \in A$. The following are equivalent:

(i) F is continuous at x_0 ($\varepsilon - \delta$ definition)
(ii) F is sequentially continuous at x_0, i.e. $\forall\ \{x_n\} \subseteq A$ such that $\{x_n\} \to x_0$, then also $\{F(x_n)\} \to F(x_0)$.

(You essentially prove this theorem in your homework problem 1.4.18., so we can skip it here.)

Theorem Let $A \subseteq \mathbb{R}^n$, $F : A \to \mathbb{R}^m$. The following are equivalent:

(i) F is continuous on A

(ii) For all $O \subseteq \mathbb{R}^m$ open, $F^{-1}(O)$ is (relatively) open in A.

This is an important and useful characterization of continous functions, so we'll recall or construct a proof in class.

Theorem $A \subseteq \mathbb{R}^n$, $f, g : A \to \mathbb{R}$; or $F, G : A \to \mathbb{R}^2$ with associated complex functions $f, g : A \to \mathbb{C}$. Then the following limit theorems hold. f, g continous at $x_0 \in A$ implies

(i) $f + g$ is continous at x_0

(ii) fg is continous at x_0

(iii) $\frac{f}{g}$ is continous at x_0 if $g(x_0) \neq 0$.

proof: Since continuity is equivalent to sequential continuity it suffices to consider all $\{x_n\} \subseteq A$ with $\{x_n\} \to x_0$, and assuming $\{f(x_n)\} \to f(x_0)$, $\{g(x_n)| \to g(x_0)$ to then show that

$$\{f(x_n) + g(x_n)\} \to f(x_0) + g(x_0)$$

$$\{f(x_n)g(x_n)\} \to f(x_0)g(x_0)$$

$$\left\{\frac{f(x_n)}{g(x_n)}\right\} \to \frac{f(x_0)}{g(x_0)}.$$

But this follows from the analogous Theorem for sequences in last week's notes, which you essentially checked in your homework 1.4.18 due today. (You could also prove this theorem with ε's and δ's.)

Corollary If $p(z)$ is a polynomial function on $\mathbb{C}$, it is continuous on all of $\mathbb{C}$. If $r(z) = \frac{p(z)}{q(z)}$ is a rational function, then it is continuous on all of $\mathbb{C}$, except at the roots of the denominator polynomial $q(z)$.

Remark So that I could appeal to your homework problem, I stated the continuity theorem first. But the same method of proof also implies the analogous limit theorem, even if the functions aren't defined or continuous at x_0.

Theorem $A \subseteq \mathbb{R}^n$, $f, g : A \to \mathbb{R}$; or $F, G : A \to \mathbb{R}^2$ with associated complex functions $f, g : A \to \mathbb{C}$. Then if

$$\lim_{x \to x_0} f(x) = L \quad \text{and} \quad \lim_{x \to x_0} g(x) = M,$$

it follows that

(i)

$$\lim_{x \to x_0} f(x) + g(x) = L + M$$

(ii) $\lim_{x \to x_0} f(x)g(x) = LM$

(iii) $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L}{M}$ provided $M \neq 0$.

Continuous functions and compactness:

(1) Theorem Let $K \subseteq \mathbb{R}^n$ compact, $F : K \to \mathbb{R}^n$ continuous. Then $F(K) \subseteq \mathbb{R}^m$ is compact.
proof:

(2) Corollary (Extreme value Theorem) Let $K \subseteq \mathbb{R}^n$ compact, $F : K \to \mathbb{R}$ continuous. Then F attains its infimum m and supremum M. In other words,

$$\exists\, x_1 \in K \text{ such that } F(x_1) = inf_{x \in K} F(x).$$
$$\exists\, x_2 \in K \text{ such that } F(x_2) = sup_{x \in K} F(x).$$

proof: By Theorem 1, $F(K)$ is compact. Because compact sets in $\mathbb{R}^n$ are bounded the infimum and supremum are finite. Because compact sets are closed, and closed sets contain all their limit points, we know that $F(K)$ contains all its limit points. Thus $F(K)$ contains its infimum and extremum. QED.

Def Let $A \subseteq \mathbb{R}^n$, $F : A \to \mathbb{R}^m$. Then F *is uniformly continuous* iff

(3) Theorem Let $K \subseteq \mathbb{R}^n$, K compact. Let $F : K \to \mathbb{R}^m$ be continous. Then F is uniformly continuous.

proof (Various approaches work, after some effort and depending on the characterization of compact sets you prefer to use. I prefer to use sequential compactness, but the text has an open cover proof.)

Connectivity and functions

The following definition is equivalent to the one we wrote down last week:

A set $A \subseteq \mathbb{R}^n$ is *"not connected"* (or *"has a disconnection"*) iff $\exists\ U, V$ relatively open subsets of A such that

$$A = U \cup V$$

$$U \neq \varnothing,\ \ V \neq \varnothing$$

$$U \cap V = \varnothing\,.$$

A set $A \subseteq \mathbb{R}^n$ is *"connected"* if it has no disconnection.

(1) Theorem Let $A \subseteq \mathbb{R}^n$ be connected, $F: A \to \mathbb{R}^m$ continuous. Then $F(A)$ is connected.
proof:

Def Let $A \subseteq \mathbb{R}^n$. A is *path connected* iff $\forall\ x, y \in A,\ \exists\ \gamma: [a, b] \to A$ continuous (a "path") such that $\gamma(a) = x, \gamma(b) = y$.

(2) Theorem If A is path connected, then A is connected.

proof: (sketch) Assume $\{U, V\}$ is a disconnection of A. Pick $x \in A$, $y \in B$ and let $\gamma: [a, b] \to A$ be a (continuous) path connecting x to y. Let

$$c := sup\{t \geq a \mid \gamma([a, t]) \subseteq A\}$$

We can show that $c > a$, $c < b$ and that $\gamma(c)$ is in neither U nor V, using the facts that U, V are both relatively open: Since U is open and $\gamma(a) \in U$, there is a $\delta > 0$ so that $\gamma([a, a + \delta]) \subseteq U$, because γ is continuous. So $c \geq a + \delta$. Similarly, $c < b$. Then, if $\gamma(c) \in U$ the fact that U is open means there is a δ>0 so that $\gamma([c - \delta, c + \delta]) \subseteq U$ which violates the definition of c. So $\gamma(c) \notin U$. Similarly, $\gamma(c) \notin V$.This is a contradiction to the assumption that $\{U, V\}$ covers A. Thus, no disconnection of A exists.

(3) Theorem If A is open and connected, then A is path connected. (Thus, for open sets, being path connected is the same as being connected.)

proof (sketch). Pick any $x_0 \in A$. Let $U = \{y \in A$ such that there is a path from x_0 to $y\}$. We can show that U is open, and that its complement $V := A \setminus U$ which is automatically closed, is also open, unless it is empty. Thus V is empty, since A is connected.

Friday September 6

1.5: complex differentiation; basic properties of analytic functions.   This will start out seeming like it's just like Calc 1, or maybe like multivariable Calculus, but we'll see that it has some deep differences.