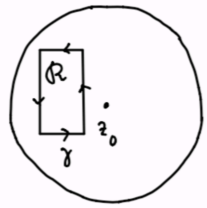
2.3 first step: improved (but local) antidifferentiation theorem:

<u>Theorem</u> Let $f : D(z_0; r) \to \mathbb{C}$ be analytic. Then $\exists\, F : D(z_0; r) \to \mathbb{C}$ such that $F' = f$ in $D(z_0; r)$.

<u>Rectangle Lemma</u> Let $f$, $D(z_0; r) = D$ be as above. Let $R = [a, b] \times [c, d] \subseteq D$ be a coordinate rectangle inside the disk. (i.e. $R = \{x + iy \mid a \le x \le b, c \le y \le d\} \subseteq D$.) Let $\gamma = \delta R$, oriented counterclockwise. Then

$$\int_\gamma f(z)\, dz = 0.$$

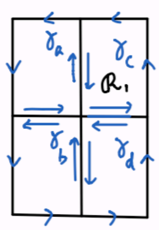



Goal: $\oint_\gamma f(z)dz = 0$

perimeter of $R = p$
diagonal length $= d$.

(If $f$ was $C^1$ we'd already know this result via Green's Theorem.)
*proof:* (Goursat):

Subdivide:

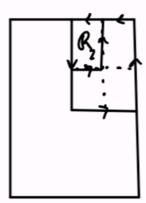



$$\int_\gamma f(z) = \int_{\gamma_a} f + \int_{\gamma_b} f + \int_{\gamma_c} + \int_{\gamma_d} \quad \text{(interior cancellation)}$$

$$\Rightarrow \left|\int_\gamma f(z)dz\right| \le \left|\int\right| + \left|\int\right| + \left|\int\right| + \left|\int\right|$$

$$\le 4\left|\int_{\gamma_1} f(z)dz\right|$$ where $\gamma_1 = \partial R_1$ has the contour integral with largest modulus

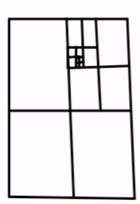
$R_1$ perimeter $= p/2$
diag $= d/2$




pick $R_2 \subset R_1$
$\partial R_2 = \gamma_2$ s.t.

$$\left|\int_\gamma f(z)dz\right| \le 4\left|\int_{\gamma_1} f(z)dz\right| \le 4^2\left|\int_{\gamma_2} f(z)dz\right|$$

$R_2$ perim $= p/2^2$
diag $d/2^2$



induct $R \supset R_1 \supset R_2 \supset \dots \supset R_k$

$$\left|\int_\gamma f(z)dz\right| \le 4^k \left|\int_{\gamma_k} f(z)dz\right|$$

$p_k$ = perimeter of $R_k = p\,2^{-k}$
$d_k$ = diag length " $= d\,2^{-k}$

$\{R_k\}$ nested, diameters $d_k \to 0$

$$\Rightarrow \bigcap_k R_k = z_0 \in R.$$ (all our rectangles are closed)

punchline: $f$ is analytic at $z_0$. Thus for $z$ near $z_0$:

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\varepsilon(z - z_0)$$

where the error function

$$\varepsilon(z - z_0) \to 0 \quad \text{as } z \to z_0.$$

Let $\epsilon > 0$. Pick $k$ such that the error satisfies

$$|\varepsilon(z - z_0)| \leq \epsilon,\ \forall\, z \in R_k\,.$$

Now estimate

$$\int_{\gamma_k} f(z)\, dz = \int_{\gamma_k} f(z_0) + f'(z_0)(z - z_0)\, dz + \int_{\gamma_k} (z - z_0)\varepsilon(z - z_0)\, dz\ .$$

By the FTC, and if $\gamma_k$ starts and ends at a point $Q$,

$$\int_{\gamma_k} f(z_0) + f'(z_0)(z - z_0)\, dz = f(z_0)z + f'(z_0)\frac{(z - z_0)^2}{2}\ \Big]_Q^Q = 0.$$

So

$$\int_{\gamma_k} f(z)\, dz = \int_{\gamma_k} (z - z_0)\varepsilon(z - z_0)\, dz$$

$$\left|\int_{\gamma_k} f(z)\, dz\right| \leq \int_{\gamma_k} |(z - z_0)\varepsilon(z - z_0)|\, |dz|$$

$$\leq d_k\, \epsilon\, p_k \ \leq \epsilon\, 2^{-k} d\, 2^{-k} p = \epsilon\, 4^{-k} p\, d.$$

And we estimate the original contour integral,

$$\left|\int_{\gamma} f(z)\, dz\right| \leq 4^k \left|\int_{\gamma_k} f(z)\, dz\right| \leq 4^k\, \epsilon\, 4^{-k} p\, d = \epsilon\, p\, d.$$

Since this estimate is true for all $\epsilon$,

$$\left|\int_{\gamma} f(z)\, dz\right| = 0$$

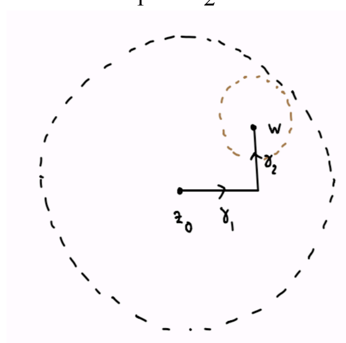
which proves the rectangle lemma.

Q.E.D.

Now complete the proof of the local antidifferentiation theorem: : This will reduce to discussion at start of class.

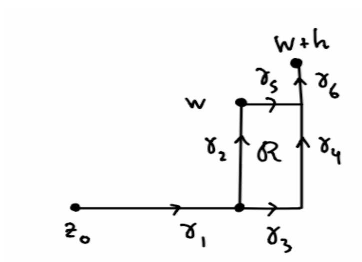
<u>Theorem</u> Let $f : \mathrm{D}(z_0; r) \to \mathbb{C}$ be analytic. Then $\exists\ F : \mathrm{D}(z_0; r) \to \mathbb{C}$ such that $F' = f$ in $\mathrm{D}(z_0; r)$.
*proof:* Let $w \in \mathrm{D}(z_0; r)$. Consider the closed rectangle $R(w)$ which has $z_0$ and $w$ as opposite corners. (This rectangle will collapse into a line segment if $w - z_0$ is purely real or imaginary. Let $\gamma_1$ be the real-direction curve from $z_0$ to $z_0 + \mathrm{Re}(w - z_0)$; let $\gamma_2$ be the imaginary direction displace from $z_0 + \mathrm{Re}\cdot(w - z_0)$ to $w$, as indicated below. Note, depending on the relative location of $z_0$ and $w$, $\gamma_1$ may move in either the positive or negative real direction; $\gamma_2$ may move in either the positive or negative imaginary direction. Define

$$F(w) = \int_{\gamma_1 + \gamma_2} f(z)\, dz.$$




To show that $F'(w) = f(w)$ we will verify the affine approximation formula with error. Let $h \in \mathrm{D}(w; r - |z_o|) \subseteq \mathrm{D}(z_0; r)$. Then, for the contours indicated below, we see that

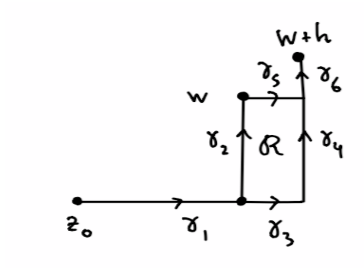
$$F(w+h) = \int_{\gamma_1 + \gamma_3 + \gamma_4 + \gamma_6} f(z)\, dz.$$




So

$$F(w+h) - F(w) = \int_{\cancel{\gamma_1} + \gamma_3 + \gamma_4 + \gamma_6} f(z)\, dz - \int_{\cancel{\gamma_1} + \gamma_2} f(z)\, dz = \int_{\gamma_3 + \gamma_4 + \gamma_6} f(z)\, dz - \int_{\gamma_2} f(z)\, dz.$$

(Repeated for lecture clarity:)




So

$$F(w+h) - F(w) = \int_{\cancel{\gamma_1} + \gamma_3 + \gamma_4 + \gamma_6} f(z)\, dz - \int_{\cancel{\gamma_1} + \gamma_2} f(z)\, dz = \int_{\underbrace{\gamma_3 + \gamma_4}_{\gamma_2 + \gamma_5} + \gamma_6} f(z)\, dz - \int_{\gamma_2} f(z)\, dz\,.$$

As the diagram indicates, the parallel curves $\gamma_2, \gamma_4$ and the parallel curves $\gamma_3, \gamma_5$ bound a rectangle (or line segment). And regardless of how this rectangle is oriented, the curves $\gamma_3 + \gamma_4$ and $\gamma_2 + \gamma_5$ have the same initial and terminal points. So by the rectangle lemma,

$$\int_{\gamma_3 + \gamma_4 - \gamma_5 - \gamma_2} f(z)\, dz = 0, \quad \text{i.e.} \quad \int_{\gamma_3 + \gamma_4} f(z)\, dz = \int_{\gamma_2 + \gamma_5} f(z)\, dz\,.$$

So

$$F(w+h) - F(w) = \int_{\cancel{\gamma_2} + \gamma_5 + \gamma_6} f(z)\, dz - \int_{\cancel{\gamma_2}} f(z)\, dz = \int_{\gamma_5 + \gamma_6} f(z)\, dz\,.$$

But this is exactly the contour integral expression we used for $F(w+h) - F(w)$ in the section 2.2 antidifferentiation theorem. And using exactly the same calculations as there,

$$\int_{\gamma_5 + \gamma_6} f(z)\, dz = \int_{\gamma_5 + \gamma_6} f(w)\, dz + \int_{\gamma_5 + \gamma_6} f(z) - f(w)\, dz$$

$$= f(w)\, h + h\, \varepsilon(h)$$

where $\varepsilon(h) \to 0$ as $h \to 0$. In other words,

$$F(w+h) = F(w) + f(w)\, h + h\, \varepsilon(h).$$

$$F'(w) = f(w).$$

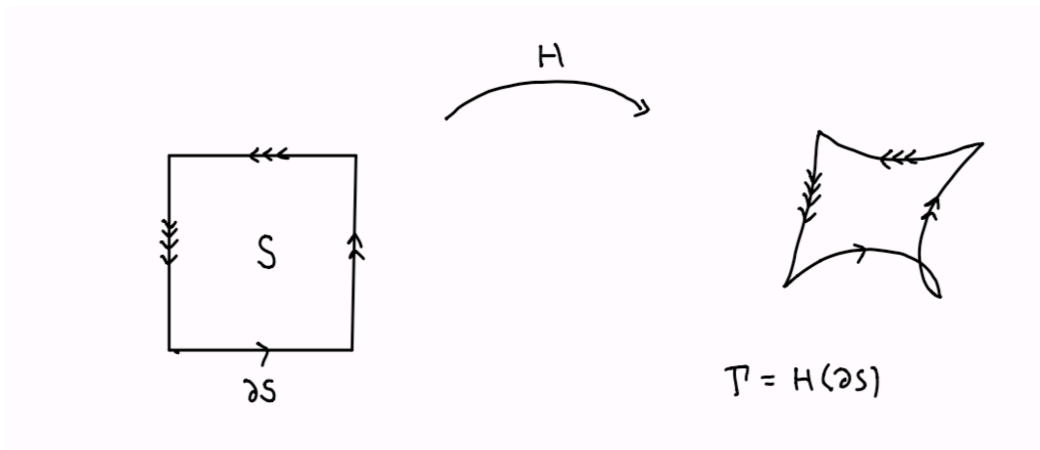
Q.E.D.

Homotopy Lemma Let $A \subseteq \mathbb{C}$ be open and connected. Let $f: A \to \mathbb{C}$ be analytic. Let

$$S = \{ (s, t) \mid 0 \le s \le 1, 0 \le t \le 1 \} \text{ and}$$
$$\delta S$$

denote the unit square and its boundary, oriented counterclockwise. Let $H : S \to A$ be continuous, with $\Gamma := H(\delta S)$ a piecewise $C^1$ contour. Then

$$\int_\Gamma f(z)\, dz = 0.$$




We will prove the homotopy lemma on the last page of this set of notes. It is the key step for the main two theorems of section 2.3:

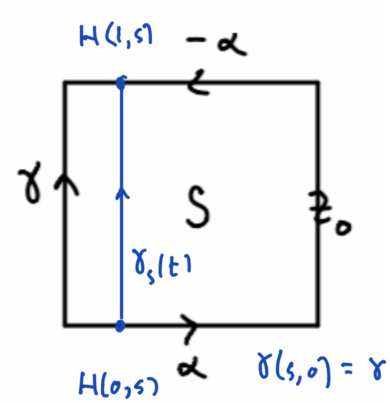
Theorem 1 Anti derivatives for analytic functions in simply connected domains: Let $A \subseteq \mathbb{C}$ be simply connected. Let $f: A \rightarrow \mathbb{C}$ analytic. Then $\exists\, F: A \rightarrow \mathbb{C}$ such that $F' = f$ in $A$.

*proof:* It suffices to prove that contour integrals are path independent, or equivalently that whenever $\gamma: [a, b] \rightarrow A$ is a piecewise $C^1$ closed curve - which we can assume is actually parameterized on the interval $[0, 1]$ - then

$$\int_\gamma f(z)\, dz = 0.$$

$\gamma$ is homotopic to $z_0$ as closed curves

~~For such a $\gamma$ there is a homotopy of $\gamma$ to a fixed point~~ $z_0 \in A$: We label the sides of the unit square by the images under this homotopy. Note that the closed curve condition means that if the lower directed segment is mapped to a curve $\alpha$, then the upper directed curve is mapped to $-\alpha$.




$\gamma(s,0) = \gamma(s,1)$ for closed paths

By the homotopy lemma

$$0 = \int_\Gamma f(z)\, dz = \cancel{\int_\alpha f(z)\, dz} + \underbrace{\int_{z_0} f(z)\, dz}_{0} - \cancel{\int_\alpha f(z)\, dz} - \int_\gamma f(z)\, dz = -\int_\gamma f(z)\, dz\,.$$

Q.E.D.

Technical note: Since the homotopy $H$ is only assumed to be continuous, the curves $\alpha$, $-\alpha$ may not be piecewise $C^1$. See the proof of the Homotopy Lemma to see how this is taken care of.

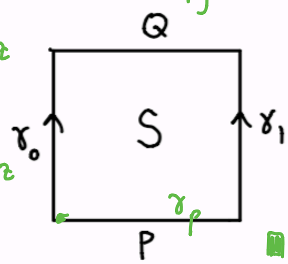
Theorem 2 Deformation Theorem Let $A \subseteq \mathbb{C}$ be open and connected (but not necessarily simply connected). Let $f: A \to \mathbb{C}$ analytic. If the two piecewise $C^1$ curves $\gamma_0, \gamma_1$ are homotopic in $A$, either with *fixed endpoints* or as *closed curves*, then

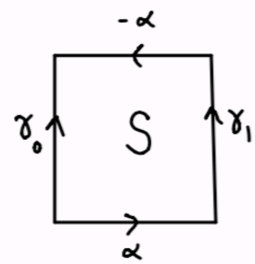
$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$$

proof: Use the homotopy lemma on these two diagrams. Again, the edges of the unit square are labeled by their images under the homotopy:

proof fixed endpoints. Homotopy lemma.

$$\cancel{\int_{\gamma_P} f(z)dz} + \int_{\gamma_1} f(z)dz + \cancel{\int_{\gamma_Q} f(z)dz} - \int_{\gamma_0} f(z)dz = 0$$



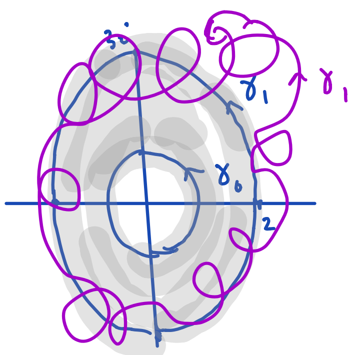

proof homotopic as closed curves

$$0 = \cancel{\int_\alpha f dz} + \int_{\gamma_1} f(z)dz - \cancel{\int_\alpha f(z)dz} - \int_{\gamma_0} f(z)dz$$

Example $\gamma_0(t) = e^{it}$ $0 \le t \le 2\pi$ , $f(z)$ analytic in $\mathbb{C} \setminus \{0\}$
$= \cos t + i \sin t.$ e.g. $f(z) = \dfrac{\cos z\, e^{5z}}{z^3}$

$\gamma_1(t) = \underset{x}{2\cos t} + \underset{y}{i\,3\sin t}$, $0 \le t \le 2\pi$.

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$




§2.2. $\oint_{\gamma_1} f(z)dz = \oint_{\gamma_0} f(z)dz$ replacement thm used Green's thm & CR eqtns.

§2.3: Homotopy in $\mathbb{C} \setminus \{0\}$ thru closed curves, from $\gamma_0$ to $\gamma_1$

$$H(s,t) = (1+s)\cos t + i(1+2s)\sin t$$

+ Theorem 2.

purple $\gamma_1$ works better with §2.3.

$$H(s,t) = \gamma_1(t)(1-s) + s\left(\frac{\gamma_1(t)}{|\gamma_1(t)|}\right)$$

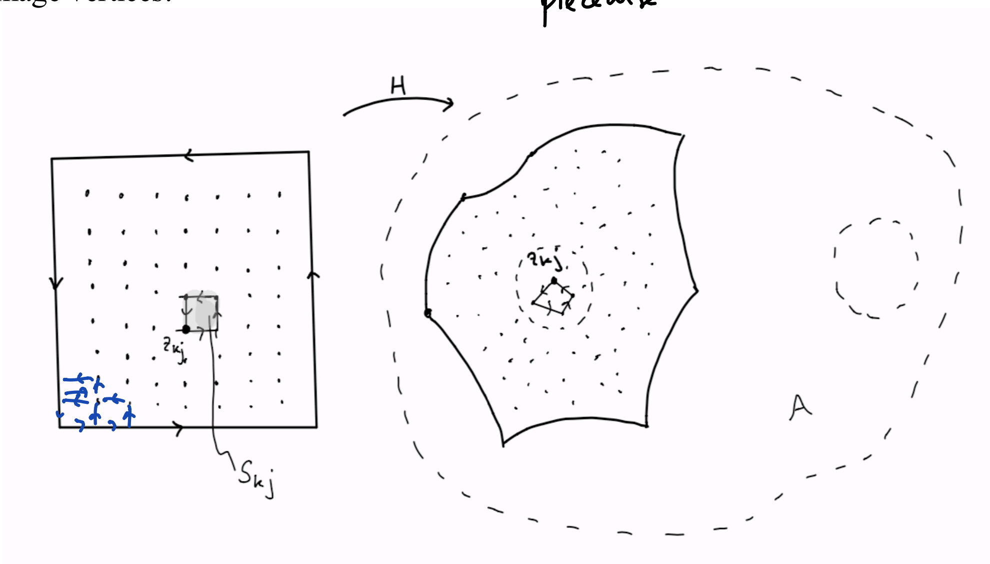
$$\gamma_1(t) \to \frac{\gamma_1(t)}{|\gamma_1(t)|}$$

then show this curve is homotopic to $\gamma_0$

*proof of the homotopy lemma:* Subdivide $S$ into $n^2$ subsquares of side lengths $\frac{1}{n}$. The dots in the diagram on the left indicate their vertices. number the squares as you would a matrix, and let $S_{kj}$ be a typical subsquare, with $z_{kj}$ be the image under the homotopy of its lower left corner. Since $H$ is continuous and $S$ is compact, the image $H(S) \subseteq A$ is compact. Write

$$H(\delta S) = \Gamma$$
$$H(\delta S_{kj}) = \Gamma_{kj}.$$

Replace any of the four subarcs of each $\Gamma_{kj}$ which are not piecewise $C^1$ with constant speed line segment paths between the image vertices.




By interior cancellation,

$$\int_{\Gamma} f(z)\,dz = \sum_{k,j} \int_{\Gamma_{kj}} f(z)\,dz.$$

Note:
1) $H(S)$ is compact, $H(S) \subseteq A$ open, so by the Positive Distance Lemma you proved in section 1.4 homework,

$$\exists\, \varepsilon > 0 \text{ such that } \forall\, z \in H(S), D(z;\varepsilon) \subseteq A.$$

2) $H$ is continuous on $S$ so $H$ is uniformly continuous. Thus for $\varepsilon$ as in (1),

$$\exists\, \delta > 0 \text{ such that } \|(s,t) - (\tilde{s},\tilde{t})\| < \delta \Rightarrow |H(s,t) - H(\tilde{s},\tilde{t})| < \varepsilon.$$

3) If $n$ is large enough so that the diagonal length $\frac{\sqrt{2}}{n}$ of the subsquares is less than $\delta$, then each

$$H(S_{kj}) \subseteq D(z_{kj};\varepsilon) \subseteq A,\ z_{kj} = H(s_k, t_j).$$

4) By the local antidifferentiation theorem in $D(z_{kj};\varepsilon)$, each

$$\int_{\Gamma_{kj}} f(z)\,dz = 0 \Rightarrow \int_{\Gamma} f(z)\,dz = 0. \qquad \text{Q.E.D.!!!}$$

Math 4200
Monday September 30 review notes

Announcements: We'll finish the proofs in Friday's notes first: homotopy lemma, antiderivatives in simply connected domains; deformation theorem. Then we'll go over today's review notes.

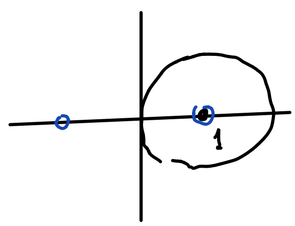
Review session today (Monday) ~~5:00-6:30~~ 4-5:30 in JTB 120. We'll go over the exam from 2011.

If you're pressed for time, you can hand in the section 2.3 homework for this week on Friday instead of Wednesday, although the exam will likely have some material from 2.3.

2.3.6 HW is about antiderivs for analytic fns in simply connected domains
Thm 2.2.4.
the only gap in that thm was the proof of path independence.
just need to quote the true proof, which
depends on the homotopy def of simply connected.

2.3.7d. $\oint_{|z-1|=1} \frac{1}{z^2-1}\,dz$

$\frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$



Exam Wednesday October 5

begin at 11:45 (5 minutes early), and end at 12:45 (5 minutes late), so that you have an hour for the exam.

closed book and closed notes

Potential Topics (we'll discuss):

<u>Complex differentiability</u> (def at a point, equivalent approximation formula, and "analytic" on a domain).

Cauchy Riemann equations $f = u + iv$
$u_x = v_y \quad u_y = -v_x$

relationship to real differentiability, i.e. $f : \mathbb{C} \to \mathbb{C}$ analytic at $z_0$ is equivalent to what for $F : \mathbb{R}^2 \to \mathbb{R}^2$ at $(x_0, y_0)$?

- $\begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix}$ is $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, rotation dilation
- F real diff'ble

consequences of definition of derivative and equivalent approximation formula

sum, product, quotient rules
chain rule
chain rule for curves • $\gamma : [a,b] \longrightarrow \mathbb{C}$, $f : \mathbb{C} \to \mathbb{C}$ analytic. $\frac{d}{dt} f(\gamma(t)) = f'(\gamma(t))\gamma'(t)$
differential map $df_{z_0}$ : $T_{z_0}\mathbb{C} \to T_{f(z_0)}\mathbb{C}$ $\quad df(\vec{h}) = f'(z_0)h$
using chain rule for curves to write CR in different coordinate systems. •

inverse function theorem •

harmonic functions and harmonic conjugates in simply-connected domains •

<u>Complex transformations</u>

polar form for complex multiplication, powers, exponentials, logarithms.

$f(z) = az + b, z^n, \ e^z, \log z, \ z^a, \ \cos z, \sin z,$ compositions

branch points, branch cuts, branch domains for root functions, logarithms, and compositions
(at least one problem)

Contour integration §2.1-2.2

for me exams are a chance for you to solidify the key ideas. So problems won't be too technical but will address key def's & results & computations

definition and computation

relation to real-variables line integrals

Green's Theorem for contour integrals around domains (including domains with holes).

contour replacement for $C^1$ analytic integrands $f(z)$, via Green's Theorem and CR equations (Section 2.2 Cauchy's Theorem)

estimates for modulus of contour integrals. $\left|\int_\gamma f(z)dz\right| \leq \int_\gamma |f(z)||dz|$

FTC

evaluation of contour integrals when the integrand is analytic, using FTC and/or contour replacement. "version 1 of Deformation in 2.2. "Cauchy's thm for domains with holes"

if $F'=f$

$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$

$\gamma:[a,b]\to\mathbb{C}$

$f$ analytic

easy to prove.

Homotopy-related ideas

homotopies

hint: any homotopies on exam will be straight-line homotopies.

fixed endpoint

of closed paths

simply-connected domains

Antiderivatives of analytic functions

equivalence to path independence §2.2.

local anti-derivative theorem, using rectangle lemma

global antiderivatives in (open) simply-connected domains, using homotopy lemma to prove path independence

Deformation Theorems via the homotopy lemma

be familiar with statements but I won't ask the proofs

for contours with fixed endpoints

for closed curves (section 2.3 Cauchy's Theorem)

Math 4200-001
Week 5 concepts and homework
2.3
Due Wednesday October 2 at start of class.

2.3 1, 3, 5, 6, 7, 9, 10. In 9b write down a homotopy from the given curve to the standard parameterization of the unit circle, in $\mathbb{C} \setminus \{0\}$, to justify your work.