Contour integrals and antiderivatives: Let $f: A \subseteq \mathbb{C} \to \mathbb{C}$ continuous, A open and connected. When does f have an antiderivative $F(z)$, i.e. $F'(z) = f(z)\ \forall\ z \in A$? (Note: antiderivatives are unique up to additive constants, since if $F' = G' = f$ on A then $(F - G)' = 0$ on A, so $F - G$ is constant because A is open and connected.)

Theorem 1 The following are equivalent, for $f: A \to \mathbb{C}$ continuous, where A is open and connected:

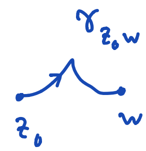
(i) $\exists\ F: A \to \mathbb{C}$ such that $F' = f$ on A

(ii) Contour integrals are *path independent*, i.e. for all choices of initial point P and terminal point Q in A,

$$\int_{\gamma_0} f(z)\ \mathrm{d}z = \int_{\gamma_1} f(z)\ \mathrm{d}z$$

whenenver γ_0, γ_1 are piecewise C^1 continuous paths that start at P and end at Q. And under this hypothesis and for any $z_0 \in A$, the function $G(w)$ defined via contour integrals as follows, satisfies $G' = f$.

$$G(w) = \int_{\gamma_{z_0 w}} f(z)\ \mathrm{d}z,$$



where $\gamma_{z_0 w}$ is any piecewise C^1 continuous path from z_0 to w

(iii) For all piecewise C^1 curves γ which have the same initial and terminal point,

$$\int_\gamma f(z)\ \mathrm{d}z = 0.$$

proof: We already know, by the FTC for contour integrals of analytic functions, that (i) implies (ii) and (iii). We'll discuss why (ii) and (iii) are equivalent. We'll use that equivalence more in section 2.3 than here in section 2.2. Then we'll do the crux of Theorem 1's proof, which is the implication (ii)⇒(i).

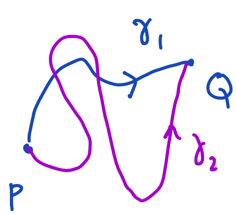
(i) ⇒ (ii): If F exists. Then $\int_{\gamma_0} f(z)dz = F(Q) - F(P)$.

In that case, $G(w)$ as above $= F(w) - F(z_0)$

(ii) ⇔ (iii) (iii) is special case of (ii), when $P = Q$, and $\int_\gamma f(z)dz = 0$ (e.g. $\gamma(t) \equiv P$, $0 \le t \le 1$)

so (ii) ⇒ (iii).

(iii) ⇒ (ii)



$$\int_{\gamma_1} f(z)dz \stackrel{?}{=} \int_{\gamma_2} f(z)dz$$

assuming contour ints over closed curves are 0 (iii)

consider $\gamma = \gamma_1 - \gamma_2$ is closed curve, start & end at P

so by (iii), $\int_\gamma f(z)dz = 0 \underset{\uparrow \text{ Fri.}}{=} \int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz$

pick this up on Wed!

(In this theorem, the domain A is open & connected.), $z_0 \in A$

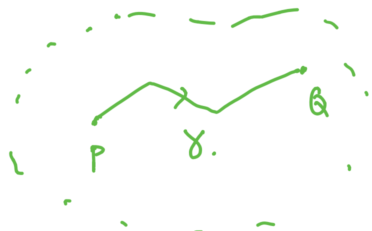
(ii) $\Rightarrow$ (i) : If contour integrals for cont. f are path ind. then

$$G(w) := \int_{\gamma_{z_0,w}} f(z)\,dz$$

is an antideriv. of f.

$\gamma_{z_0,w}$ is any p.w. C^1 curve from z_0 to w.

- point: If A is open and connected then it's pathwise connected

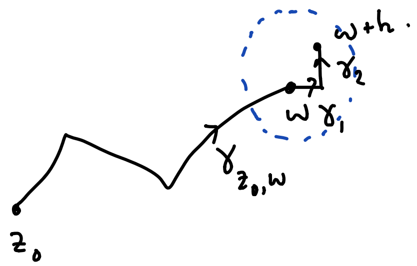


means $\forall$ P, Q in A $\exists$ a continuous curve $\gamma : [a,b] \to A$ s.t. $\gamma(a) = P$, $\gamma(b) = Q$

(γ's exist which are piecewise line segments)

(see appendix)

Need to show $G'(w) = f(w)$ $\forall\, w \in A$.



$$G(w+h) = \int_{\gamma_{z_0,w} + \gamma_1 + \gamma_2} f(z)\,dz$$

real displacement from w to $w+h$. imag

$$G(w+h) = G(w) + \int_{\gamma_1+\gamma_2} f(z)\,dz$$

$$= G(w) + \int_{\gamma_1+\gamma_2} f(w)\,dz + \underbrace{\int_{\gamma_1+\gamma_2} f(z) - f(w)\,dz}_{\text{error}}$$

const.

$$G(w+h) = G(w) + f(w)z\Big]_w^{w+h} + \text{error}.$$

FTC.

$$\boxed{G(w+h) = G(w) + f(w)h + \underbrace{\text{error}}_{\varepsilon(h)h.}}$$

need to show $\varepsilon(h) \to 0$

est. $$|\varepsilon(h)| = \frac{1}{|h|}\left|\int_{\gamma_1+\gamma_2} f(z) - f(w)\,dz\right|$$

$$\leq \frac{1}{|h|}\int_{\gamma_1+\gamma_2} \underbrace{|f(z)-f(w)|}_{\geq\!\!/\wedge}\,|dz|$$

$$\max\{|f(z)-f(w)| \text{ s.t. } |z-w| \leq |h|\}$$

$$\leq \frac{1}{\cancel{|h|}}\,\text{"max"}\{|f(z)-f(w)| \text{ s.t. } |z-w| \leq |h|\}\, 2\cancel{|h|}$$

$\to 0$ because f is continuous

So $G'(w) = f(w)$ ∎

"no holes"

Theorem 2 If A is open and simply connected. Let $f: A \to \mathbb{C}$ be analytic and C^1. Then condition (ii) of Theorem 1 holds by Cauchy's Theorem, so we see that f has an antiderivative G given by

path ind.

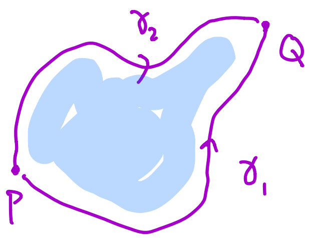
$$G(w) = \int_{\gamma} f(z)\, \mathrm{d}z$$

where we fix any $z_0 \in A$ and let γ be any piecewise C^1 curve in A connecting z_0 to w.

proof: Explain why the path-independence condition (ii) of Theorem 1 holds. (This explanation is somewhat imprecise, but we'll make everything rigorous in section 2.3)

"proof" of path ind. Consider $P, Q \in A$, paths from P to Q

Case 1

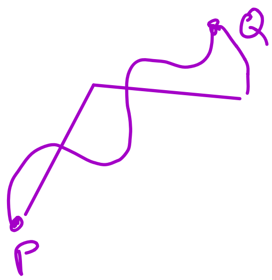


$\vec{\gamma}_1 - \vec{\gamma}_2$ bounds a region inside A (A had no holes)

Green's thm $\Rightarrow \int_{\gamma_1 - \gamma_2} f(z)\,dz = 0$

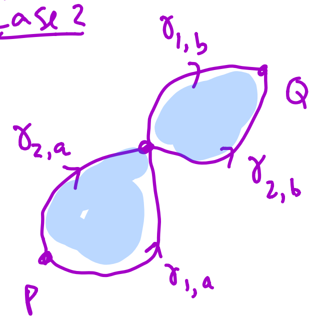
$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

induct!



Case 2

curves cross once.



$$\int_{\gamma_{1,a} - \gamma_{2,a}} f(z)\,dz = 0, \quad \int_{\gamma_{2,b} - \gamma_{1,b}} f(z)\,dz = 0.$$

$$\int_{\gamma_{1,a}} f = \int_{\gamma_{2,a}} f \; ; \; \int_{\gamma_{1,b}} f = \int_{\gamma_{2,b}} f$$

$$\int_{\gamma_{1,a}} f + \int_{\gamma_{1,b}} f = \int_{\gamma_{2,a}} f + \int_{\gamma_{2,b}}$$

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$

Appendix: Connected domains, path connected domains, simply connected domains: Some Chapter 1 analysis background material we need now:

Recall that a domain $A \subseteq \mathbb{C}$ is called *connected* iff there is no disconnection of A into disjoint (relatively) open and non-empty subsets U, V i.e. such that

$$A = U \cup V$$
$$U \cap V = \varnothing.$$

If we restrict to open domains A, then subsets U, V that are relatively open are actually open.

There is a related definition:

Def A subset $A \subseteq \mathbb{C}$ is called *path connected* iff $\forall\ P, Q \in A$, there exists a continuous path $\gamma : [a, b] \rightarrow A$ such that $\gamma(a) = P, \gamma(b) = Q$.

Theorem Let $A \subseteq \mathbb{C}$ be open. Then A is connected if and only if A is path connected. Furthermore, if A is connected then there are piecewise C^1 paths connecting all possible pairs of points in A. (Analogous theorem holds in $\mathbb{R}^n$.)

proof: $\Rightarrow$: Let A be connected and open. We will show it is path connected, with piecewise C^1 paths. Pick any base point $z_0 \in A$. Define U to be the set of points that can be connected to z_0 with a piecewise C^1 path. U is non-empty since $\mathrm{D}(z_0; r) \subseteq U$ as long as r is small enough so that the disk is in A. In fact, for all $z \in \mathrm{D}(z_0; r)$ we can use the straight-line paths

$$\gamma(t) = z_0 + t(z - z_0), \quad 0 \leq t \leq 1$$

to connect z_0 to z.

The proof that U is open is analogous: Let $w \in U$ and let γ be a piecewise C^1 path connecting z_0 to w. Then for $\mathrm{D}(w, r) \subseteq A$ and

$$\gamma_1(t) = w + t(z - w), \quad 0 \leq t \leq 1$$

the combined path $\gamma + \gamma_1$ is a piecewise C^1 path connecting z_0 to w. Thus U is open.

Finally, the complement of $V := A \setminus U$ is open by a similar argument: Let $z_1 \in V$, $\mathrm{D}(z_1; r) \subseteq A$. Then $\mathrm{D}(z_1, r) \subseteq V$ as well, since if $\exists\ w \in U \cap \mathrm{D}(z_1; r)$ there is a piecewise C^1 path γ from z_0 to w, and letting

$$\gamma_2(t) = w + t(z_1 - w), \quad 0 \leq t \leq 1,$$

the path $\gamma + \gamma_2$ would connect z_0 to z_1. Thus, since A is connected, we must have that $V = A \setminus U$ is empty.

reverse implication: Assume A is path connected. Then A is connected: Let $A = U \cup V$ with U, V open, U non-empty, and $U \cap V = \emptyset$. We will show V is empty. If not, pick $P \in U$, $Q \in V$, and let

$$\gamma : [a, b] \to \mathbb{C}$$

be a continuous path connecting P to Q, i.e. $\gamma(a) = P$, $\gamma(b) = Q$. Let $T \in [a, b]$ be defined by

$$T := sup\{t \in [a, b] \mid \gamma([a, t]) \subseteq U\}$$

Because U is open, $T > a$. If $T = b$ we would have $Q \in U$ which would be a contradiction. But if $a < T < b$ then $\gamma(T)$ is in neither U nor V: If $\gamma(T) \in U$ then by continuity and U open, there exists $\delta > 0$ so that $\gamma([T, T + \delta)) \subseteq U$, hence $\gamma([a, T + \delta)) \subseteq U$, contradicting the definition of T. Similarly, if $\gamma(T) \in V$, continuity of γ and V open implies there exists $\delta > 0$ so that $\gamma([T - \delta, T] \subseteq V$, another contradiction. Thus T can't exist, and V must be empty.

Math 4200
Wednesday September 25

2.2-2.3 Antiderivatives for analytic functions and Cauchy's Theorem: Finish 2.2 discussion based on Green's Theorem and begin 2.3 discussion which is more general and more rigorous.

Announcements: We'll finish Monday's notes first, and they will lead naturally into today's section 2.3 refinements

- I didn't finish grading last week's HW yet... Friday.
- next HW at end of today's notes: §2.3
- exam 1 next Wed.
  thru at least §2.2.
  probably thru §2.3.

plan: you can do some of these already - using §2.2.
Hopefully all are accessible after Friday class &
I can answer questions on Monday.

After finishing Monday's notes we will have carefully proven:

<u>Theorem</u> If A is a connected open subset of $\mathbb{C}$, $f: A \to \mathbb{C}$ continuous, then contour integrals $\int_\gamma f(z)\, dz$ are path independent if and only if $\exists$ an antiderivative $F(z)$ to $f(z)$ (and F can be defined with contour integrals).

✓

We have less carefully proven that

<u>Theorem</u> If A is open and *simply connected*, $f: A \to \mathbb{C}$ analytic and C^1, then $\int_\gamma f(z)\, dz$ are path-independent, so there exist antiderivatives $F(z)$.

<u>issues:</u>

(i) We defined a domain to be *simply connected* to mean that it has no holes. This is visually appealing but imprecise and hard to describe analytically.

(ii) We did not show path independence for all piecewise C^1 paths - we just drew a few pictures which are not representative of all possible configurations: we assumed they crossed in such a way as to create a finite number of subdomains on which to apply Green's Theorem.

(iii) We had to assume $f \in C^1$ for Green's Theorem, whereas it turns out we will only need that f is analytic on A for Cauchy's Theorem and antidifferentiation theorems (and this is useful to know).

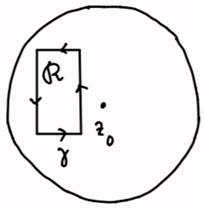
<u>The goal of section 2.3</u> is to deal with these issues, and also to prove precise and stronger versions of the <u>Deformation Theorem</u>, about when you can change the contour curves for $\int_\gamma f(z)\, dz$ without changing the actual value of the contour integrals. Along the way we introduce the notion of *homotopy,* key to many areas of mathematics, especially ones that use *algebraic topology*. (See Wikipedia.)

2.3 first step: improved (but local) antidifferentiation theorem:

<u>Theorem</u> Let $f: D(z_0; r) \to \mathbb{C}$ be analytic. Then $\exists\, F: D(z_0; r) \to \mathbb{C}$ such that $F' = f$ in $D(z_0; r)$.

<u>Rectangle Lemma</u> Let f, $D(z_0; r) = D$ be as above. Let $R = [a, b] \times [c, d] \subseteq D$ be a coordinate rectangle inside the disk. (i.e. $R = \{x + i y \mid a \le x \le b, c \le y \le d\} \subseteq D$.) Let $\gamma = \delta R$, oriented counterclockwise. Then

$$\int_\gamma f(z)\, dz = 0.$$

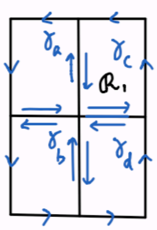


Goal: $\oint_\gamma f(z)dz = 0$

perimeter of $R = p$
diagonal length $= d$.

(If f was C^1 we'd already know this result via Green's Theorem.)
proof: (Goursat):

Subdivide:

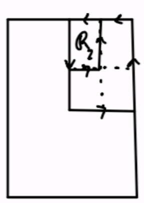


$$\int_\gamma f(z) = \int_{\gamma_a} f + \int_{\gamma_b} f + \int_{\gamma_c} + \int_{\gamma_d} \quad \text{(interior cancellation)}$$

$$\Rightarrow \left|\int_\gamma f(z)dz\right| \le |\textstyle\int| + |\int| + |\int| + |\int|$$

$$\le 4\left|\int_{\gamma_1} f(z)dz\right|$$ where $\gamma_1 = \partial R_1$ has the contour integral with largest modulus

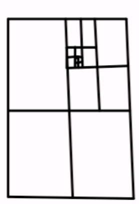
R_1 perimeter $= P/2$
diag $= d/2$



pick $R_2 \subset R_1$
$\partial R_2 = \gamma_2$ s.t.

$$\left|\int_\gamma f(z)dz\right| \le 4\left|\int_{\gamma_1} f(z)dz\right| \le 4^2\left|\int_{\gamma_2} f(z)dz\right|$$

R_2 perim $= P/2^2$
diag $d/2^2$



induct $R \supset R_1 \supset R_2 \supset .. \supset R_k$

$$\left|\int_\gamma f(z)dz\right| \le 4^k\left|\int_{\gamma_k} f(z)dz\right|$$

p_k = perimeter of $R_k = p\,2^{-k}$
d_k = diag length " $= d\,2^{-k}$

$\{R_k\}$ nested, diameters $d_k \to 0$

$\Rightarrow \bigcap_k R_k = z_0 \in R.$ (all our rectangles are closed) punchline: f is analytic at z_0. Thus for z near z_0:

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\varepsilon(z - z_0)$$

where the error function

$$\varepsilon(z - z_0) \to 0 \quad \text{as } z \to z_0.$$

Let $\epsilon > 0$. Pick k such that the error satisfies

$$|\varepsilon(z - z_0)| \le \epsilon, \ \forall\, z \in R_k .$$

Now estimate

$$\int_{\gamma_k} f(z)\, dz = \int_{\gamma_k} f(z_0) + f'(z_0)(z - z_0)\, dz + \int_{\gamma_k} (z - z_0)\varepsilon(z - z_0)\, dz \ .$$

By the FTC, and if γ_k starts and ends at a point Q,

$$\int_{\gamma_k} f(z_0) + f'(z_0)(z - z_0)\, dz = f(z_0)z + f'(z_0)\frac{(z - z_0)^2}{2} \Big]_Q^Q = 0.$$

So

$$\int_{\gamma_k} f(z)\, dz = \int_{\gamma_k} (z - z_0)\varepsilon(z - z_0)\, dz$$

$$\left| \int_{\gamma_k} f(z)\, dz \right| \le \int_{\gamma_k} |(z - z_0)\varepsilon(z - z_0)|\, |dz|$$

$$\le d_k\, \epsilon\, p_k \ \le \epsilon\, 2^{-k} d\, 2^{-k} p = \epsilon\, 4^{-k} p\, d.$$

And we estimate the original contour integral,

$$\left| \int_{\gamma} f(z)\, dz \right| \le 4^k \left| \int_{\gamma_k} f(z)\, dz \right| \le 4^k \epsilon\, 4^{-k} p\, d = \epsilon\, p\, d.$$

Since this estimate is true for all ϵ,

$$\left| \int_{\gamma} f(z)\, dz \right| = 0$$

which proves the rectangle lemma.

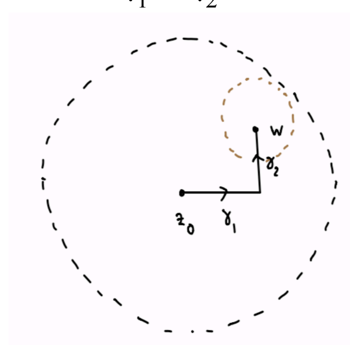
Q.E.D.

Now complete the proof of the local antidifferentiation theorem: : This will reduce to discussion at start of class.

<u>Theorem</u> Let $f: \mathrm{D}(z_0; r) \to \mathbb{C}$ be analytic. Then $\exists\, F: \mathrm{D}(z_0; r) \to \mathbb{C}$ such that $F' = f$ in $\mathrm{D}(z_0; r)$.

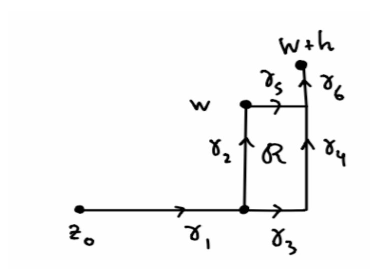
proof: Let $w \in \mathrm{D}(z_0; r)$. Consider the closed rectangle $R(w)$ which has z_0 and w as opposite corners. (This rectangle will collapse into a line segment if $w - z_0$ is purely real or imaginary. Let γ_1 be the real-direction curve from z_0 to $z_0 + \mathrm{Re}(w - z_0)$; let γ_2 be the imaginary direction displace from $z_0 + \mathrm{Re}\cdot(w - z_0)$ to w, as indicated below. Note, depending on the relative location of z_0 and w, γ_1 may move in either the positive or negative real direction; γ_2 may move in either the positive or negative imaginary direction. Define

$$F(w) = \int_{\gamma_1 + \gamma_2} f(z)\, dz.$$



To show that $F'(w) = f(w)$ we will verify the affine approximation formula with error. Let $h \in \mathrm{D}(w; r - |z_o|) \subseteq \mathrm{D}(z_0; r)$. Then, for the contours indicated below, we see that

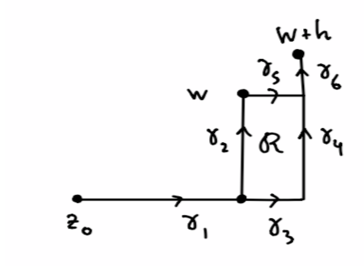
$$F(w+h) = \int_{\gamma_1 + \gamma_3 + \gamma_4 + \gamma_6} f(z)\, dz.$$



So

$$F(w+h) - F(w) = \int_{\gamma_1 + \gamma_3 + \gamma_4 + \gamma_6} f(z)\, dz - \int_{\gamma_1 + \gamma_2} f(z)\, dz = \int_{\gamma_3 + \gamma_4 + \gamma_6} f(z)\, dz - \int_{\gamma_2} f(z)\, dz.$$

(Repeated for lecture clarity:)



So

$$F(w+h) - F(w) = \int_{\gamma_1+\gamma_3+\gamma_4+\gamma_6} f(z)\,dz - \int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_3+\gamma_4+\gamma_6} f(z)\,dz - \int_{\gamma_2} f(z)\,dz\,.$$

As the diagram indicates, the parallel curves γ_2, γ_4 and the parallel curves γ_3, γ_5 bound a rectangle (or line segment). And regardless of how this rectangle is oriented, the curves $\gamma_3+\gamma_4$ and $\gamma_2+\gamma_5$ have the same initial and terminal points. So by the rectangle lemma,

$$\int_{\gamma_3+\gamma_4-\gamma_5-\gamma_2} f(z)\,dz = 0, \quad \text{i.e.} \quad \int_{\gamma_3+\gamma_4} f(z)\,dz = \int_{\gamma_2+\gamma_5} f(z)\,dz\,.$$

So

$$F(w+h) - F(w) = \int_{\gamma_2+\gamma_5+\gamma_6} f(z)\,dz - \int_{\gamma_2} f(z)\,dz = \int_{\gamma_5+\gamma_6} f(z)\,dz\,.$$

But this is exactly the contour integral expression we used for $F(w+h) - F(w)$ in the section 2.2 antidifferentiation theorem. And using exactly the same calculations as there,

$$\int_{\gamma_5+\gamma_6} f(z)\,dz = \int_{\gamma_5+\gamma_6} f(w)\,dz + \int_{\gamma_5+\gamma_6} f(z) - f(w)\,dz$$

$$= f(w)\,h + h\,\varepsilon(h)$$

where $\varepsilon(h) \to 0$ as $h \to 0$. In other words,

$$F(w+h) = F(w) + f(w)\,h + h\,\varepsilon(h).$$

$$F'(w) = f(w).$$

Q.E.D.

Math 4200-001
Week 5 concepts and homework
2.3
Due Wednesday October 2 at start of class.

2.3 1, 3, 5, 6, 7, 9, 10. In 9b write down a homotopy from the given curve to the standard parameterization of the unit circle, in $\mathbb{C} \setminus \{0\}$, to justify your work.