Math 4200
Wednesday September 25

2.2-2.3 Antiderivatives for analytic functions and Cauchy's Theorem: Finish 2.2 discussion based on Green's Theorem and begin 2.3 discussion which is more general and more rigorous.

<u>Announcements:</u> We'll finish Monday's notes first, and they will lead naturally into today's section 2.3 refinements

After finishing Monday's notes we will have carefully proven:

<u>Theorem</u> If A is a connected open subset of $\mathbb{C}$, $f: A \rightarrow \mathbb{C}$ continuous, then contour integrals $\int_{\gamma} f(z)\, dz$ are path independent if and only if $\exists$ an antiderivative $F(z)$ to $f(z)$ (and F can be defined with contour integrals).

We have less carefully proven that

<u>Theorem</u> If A is open and *simply connected*, $f: A \rightarrow \mathbb{C}$ analytic and C^1, then $\int_{\gamma} f(z)\, dz$ are path-independent, so there exist antiderivatives $F(z)$.

<u>issues:</u>

(i) We defined a domain to be *simply connected* to mean that it has no holes. This is visually appealing but imprecise and hard to describe analytically.

(ii) We did not show path independence for all piecewise C^1 paths - we just drew a few pictures which are not representative of all possible configurations: we assumed they crossed in such a way as to create a finite number of subdomains on which to apply Green's Theorem.

(iii) We had to assume $f \in C^1$ for Green's Theorem, whereas it turns out we will only need that f is analytic on A for Cauchy's Theorem and antidifferentiation theorems (and this is useful to know).

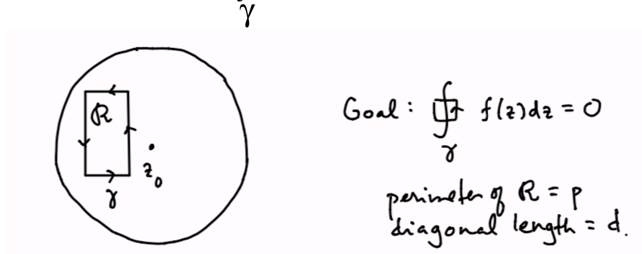
<u>The goal of section 2.3</u> is to deal with these issues, and also to prove precise and stronger versions of the <u>Deformation Theorem</u>, about when you can change the contour curves for $\int_{\gamma} f(z)\, dz$ without changing the actual value of the contour integrals. Along the way we introduce the notion of *homotopy,* key to many areas of mathematics, especially ones that use *algebraic topology*. (See Wikipedia.)

2.3 first step: improved (but local) antidifferentiation theorem:

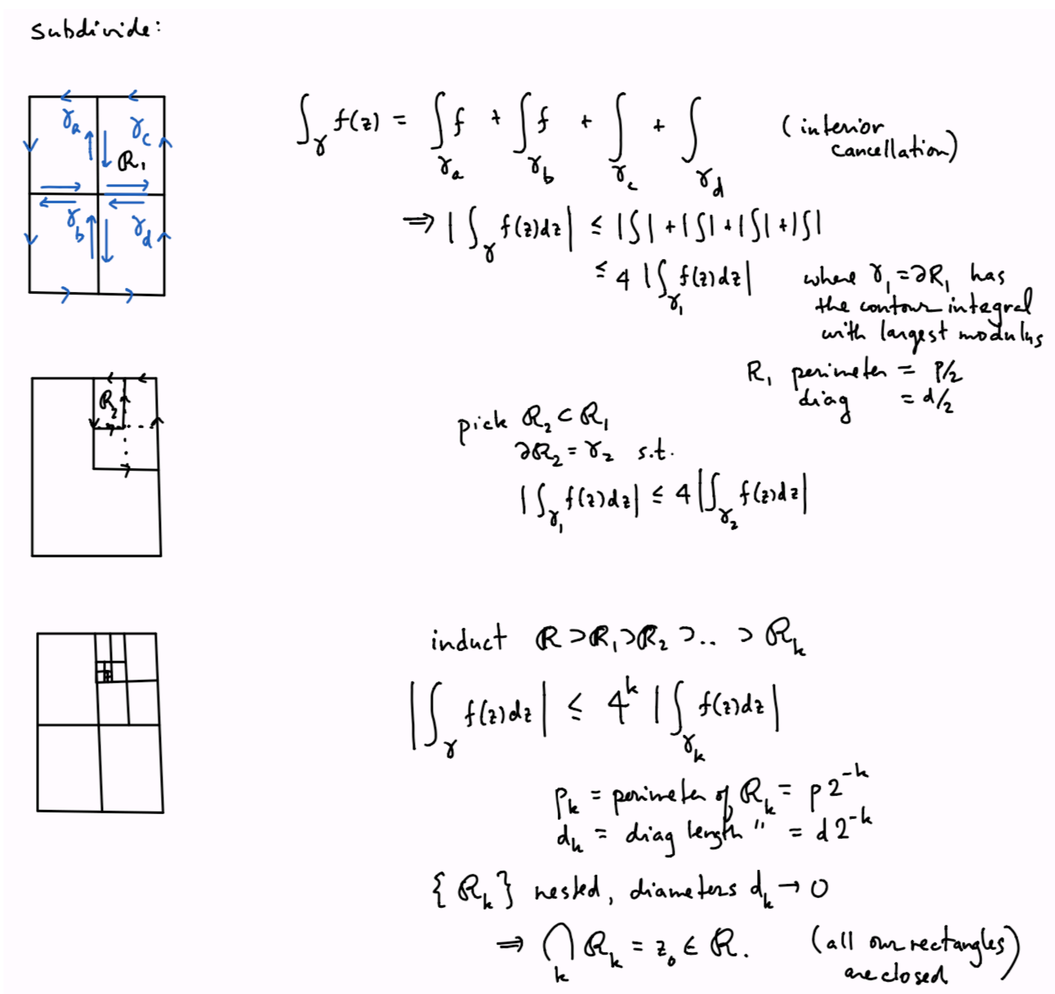
<u>Theorem</u> Let $f : D(z_0; r) \to \mathbb{C}$ be analytic. Then $\exists\, F : D(z_0; r) \to \mathbb{C}$ such that $F' = f$ in $D(z_0; r)$.

<u>Rectangle Lemma</u> Let f, $D(z_0; r) = D$ be as above. Let $R = [a, b] \times [c, d] \subseteq D$ be a coordinate rectangle inside the disk. (i.e. $R = \{x + i y \mid a \leq x \leq b, c \leq y \leq d\} \subseteq D$.) Let $\gamma = \delta R$, oriented counterclockwise. Then

$$\int_\gamma f(z)\, dz = 0.$$



(If f was C^1 we'd already know this result via Green's Theorem.)
proof: (Goursat):



punchline: f is analytic at z_0. Thus for z near z_0:

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\varepsilon(z - z_0)$$

where the error function

$$\varepsilon(z - z_0) \to 0 \quad \text{as } z \to z_0.$$

Let $\epsilon > 0$. Pick k such that the error satisfies

$$|\varepsilon(z - z_0)| \le \epsilon,\ \forall\, z \in R_k.$$

Now estimate

$$\int_{\gamma_k} f(z)\, dz = \int_{\gamma_k} f(z_0) + f'(z_0)(z - z_0)\, dz + \int_{\gamma_k} (z - z_0)\varepsilon(z - z_0)\, dz.$$

By the FTC, and if γ_k starts and ends at a point Q,

$$\int_{\gamma_k} f(z_0) + f'(z_0)(z - z_0)\, dz = f(z_0)z + f'(z_0)\frac{(z - z_0)^2}{2}\Big]_Q^Q = 0.$$

So

$$\int_{\gamma_k} f(z)\, dz = \int_{\gamma_k} (z - z_0)\varepsilon(z - z_0)\, dz$$

$$\left|\int_{\gamma_k} f(z)\, dz\right| \le \int_{\gamma_k} |(z - z_0)\varepsilon(z - z_0)|\, |dz|$$

$$\le d_k\, \epsilon\, p_k \le \epsilon\, 2^{-k} d\, 2^{-k} p = \epsilon\, 4^{-k} p\, d.$$

And we estimate the original contour integral,

$$\left|\int_{\gamma} f(z)\, dz\right| \le 4^k \left|\int_{\gamma_k} f(z)\, dz\right| \le 4^k\, \epsilon\, 4^{-k} p\, d = \epsilon\, p\, d.$$

Since this estimate is true for all ϵ,

$$\left|\int_{\gamma} f(z)\, dz\right| = 0$$

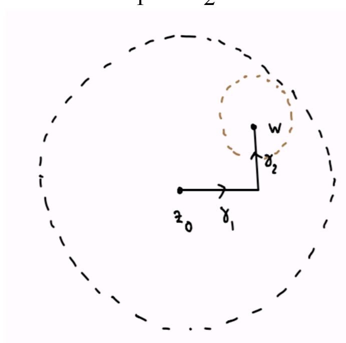
which proves the rectangle lemma.

Q.E.D.

Now complete the proof of the local antidifferentiation theorem:

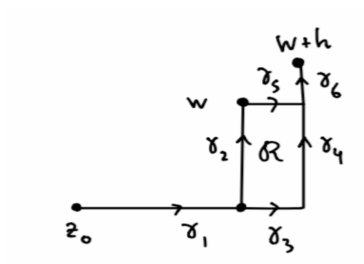
<u>Theorem</u> Let $f: \mathrm{D}(z_0; r) \to \mathbb{C}$ be analytic. Then $\exists\ F: \mathrm{D}(z_0; r) \to \mathbb{C}$ such that $F' = f$ in $\mathrm{D}(z_0; r)$.
proof: Let $w \in \mathrm{D}(z_0; r)$. Consider the closed rectangle $R(w)$ which has z_0 and w as opposite corners. (This rectangle will collapse into a line segment if $w - z_0$ is purely real or imaginary. Let γ_1 be the real-direction curve from z_0 to $z_0 + \mathrm{Re}(w - z_0)$; let γ_2 be the imaginary direction displace from $z_0 + \mathrm{Re}\cdot(w - z_0)$ to w, as indicated below. Note, depending on the relative location of z_0 and w, γ_1 may move in either the positive or negative real direction; γ_2 may move in either the positive or negative imaginary direction. Define

$$F(w) = \int_{\gamma_1 + \gamma_2} f(z)\, dz.$$



To show that $F'(w) = f(w)$ we will verify the affine approximation formula with error. Let $h \in \mathrm{D}(w; r - |z_o|) \subseteq \mathrm{D}(z_0; r)$. Then, for the contours indicated below, we see that

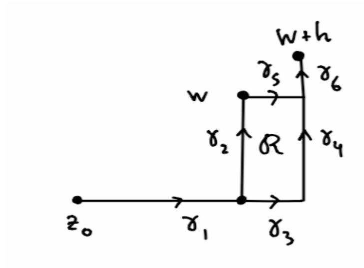
$$F(w+h) = \int_{\gamma_1 + \gamma_3 + \gamma_4 + \gamma_6} f(z)\, dz.$$



So

$$F(w+h) - F(w) = \int_{\gamma_1 + \gamma_3 + \gamma_4 + \gamma_6} f(z)\, dz - \int_{\gamma_1 + \gamma_2} f(z)\, dz = \int_{\gamma_3 + \gamma_4 + \gamma_6} f(z)\, dz - \int_{\gamma_2} f(z)\, dz.$$

(Repeated for lecture clarity:)



So

$$F(w+h) - F(w) = \int_{\gamma_1+\gamma_3+\gamma_4+\gamma_6} f(z)\,dz - \int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_3+\gamma_4+\gamma_6} f(z)\,dz - \int_{\gamma_2} f(z)\,dz .$$

As the diagram indicates, the parallel curves γ_2, γ_4 and the parallel curves γ_3, γ_5 bound a rectangle (or line segment). And regardless of how this rectangle is oriented, the curves $\gamma_3 + \gamma_4$ and $\gamma_2 + \gamma_5$ have the same initial and terminal points. So by the rectangle lemma,

$$\int_{\gamma_3+\gamma_4-\gamma_5-\gamma_2} f(z)\,dz = 0, \quad \text{i.e.} \quad \int_{\gamma_3+\gamma_4} f(z)\,dz = \int_{\gamma_2+\gamma_5} f(z)\,dz .$$

So

$$F(w+h) - F(w) = \int_{\gamma_2+\gamma_5+\gamma_6} f(z)\,dz - - \int_{\gamma_2} f(z)\,dz = \int_{\gamma_5+\gamma_6} f(z)\,dz .$$

But this is exactly the contour integral expression we used for $F(w+h) - F(w)$ in the section 2.2 antidifferentiation theorem. And using exactly the same calculations as there,

$$\int_{\gamma_5+\gamma_6} f(z)\,dz = \int_{\gamma_5+\gamma_6} f(w)\,dz + \int_{\gamma_5+\gamma_6} f(z) - f(w)\,dz$$
$$= f(w)\,h + h\,\varepsilon(h)$$

where $\varepsilon(h) \to 0$ as $h \to 0$. In other words,

$$F(w+h) = F(w) + f(w)\,h + h\,\varepsilon(h).$$
$$F'(w) = f(w).$$

Q.E.D.