Math 4200
Monday September 23

2.1-2.2 Path independence of contour integrals, and antiderivatives for analytic functions on open, simply-connected domains.

Announcements:

Recall:

The connection between contour integrals and Calc 3 line integrals:

Let $A \subseteq \mathbb{C}$ open, $f: A \to \mathbb{C}$ continuous, $\gamma: [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ a C^1 curve. write

$$\gamma(t) = x(t) + i\,y(t),$$
$$f(z) = u(x, y) + i\,v(x, y).$$

Then

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$$

$$= \int_a^b (u(x(t), y(t)) + i\,v(x(t), y(t)))(x'(t) + i\,y'(t))\,\mathrm{d}t$$

$$= \int_a^b u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)dt \; + i\int_a^b v(x(t), y(t))x'(t) + u(x(t), y(t))y'(t)dt$$

$$= \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy.$$

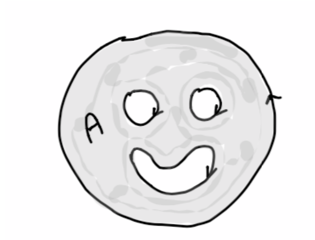
Note added (and we used this on Friday) You recover the correct real line integrals if you just substitute $f = u + i\,v$, $dz = dx + i\,dy$ into the countour integral expression:

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma (u + i\,v)\,(dx + i\,dy)$$

Recall: we talked about the fact that Green's Theorem holds for domains with and without holes, and that this leads to a corresponding theorem about sums of contour integrals for analytic functions around the boundaries of such domains. If f is analytic and C^1 in and open set containing the closure of a bounded domain A which has piecewise C^1 boundary curves, then the sum of the appropriately oriented contour integrals around the boundary is zero. Although Green's Theorem and hence Cauchy's Theorem can be proven directly for such domains (see last page appendix in today's notes, which I'd meant to include in Friday's notes), we can also recover this fact from the special case of these theorems in domains without holes. Let's do that.

Revisit from Friday: Modify the domain below to reproduce the result we quoted on Friday for domains with and without holes, from Green's Theorem just for domains without holes:

$$\int_{\delta A} P\,dx + Q\,dy = \iint_A Q_x - P_y\,dA$$



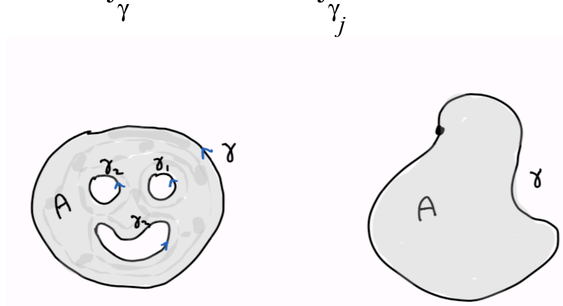
Cauchy's Theorem (section 2.2 version)

Let A be a bounded domain with piecewise C^1 boundary. Let f be analytic and C^1 on the closure $\overline{A}$. If A is simply-connected (i.e. has no holes) and the boundary of A is a single simple closed curve γ, then regardless of the direction of γ,

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

If A *does* have a finite number of holes inside γ, bounded by piecewise C^1 curves $\gamma_1, \gamma_2, \ldots \gamma_n$ and if we orient *all* boundary contours to be counterclockwise, then

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{j=1}^{n} \int_{\gamma_j} f(z)\,\mathrm{d}z$$



Example: Let $\gamma = \{z \in \mathbb{C} \mid |z| = 2\}$, oriented counterclockwise. Find the value of

$$\int_\gamma \frac{4z}{z^2 - 1}\, dz.$$

Hint:

$$\frac{4z}{z^2 - 1} = \frac{2}{z-1} + \frac{2}{z+1}$$

Contour integrals and antiderivatives: Let $f: A \subseteq \mathbb{C} \to \mathbb{C}$ continuous, A open and connected. When does f have an antiderivative $F(z)$, i.e. $F'(z) = f(z) \, \forall \, z \in A$? (Note: antiderivatives are unique up to additive constants, since if $F' = G' = f$ on A then $(F - G)' = 0$ on A, so $F - G$ is constant because A is open and connected.

Theorem 1 The following are equivalent, for $f: A \to \mathbb{C}$ continuous, where A is open and connected:

(i) $\exists \, F: A \to \mathbb{C}$ such that $F' = f$ on A

(ii) Contour integrals are *path independent*, i.e. for all choices of initial point P and terminal point Q in A,

$$\int_{\gamma_0} f(z) \, \mathrm{d}z = \int_{\gamma_1} f(z) \, \mathrm{d}z$$

whenenver γ_0, γ_1 are piecewise C^1 continuous paths that start at P and end at Q. And under this hypothesis and for any $z_0 \in A$, the function $G(w)$ defined via contour integrals as follows, satisfies $G' = f$.

$$G(w) = \int_{\gamma_{z_0 w}} f(z) \, \mathrm{d}z,$$

where $\gamma_{z_0 w}$ is any piecewise C^1 continuous path from z_0 to w

(iii) For all piecewise C^1 curves γ which have the same initial and terminal point,

$$\int_{\gamma} f(z) \, \mathrm{d}z = 0.$$

proof: We already know, by the FTC for contour integrals of analytic functions, that (i) implies (ii) and (iii). We'll discuss why (ii) and (iii) are equivalent. We'll use that equivalence more in section 2.3 than here in section 2.2. Then we'll do the crux of Theorem 1's proof, which is the implication (ii)⇒(i).

Theorem 2 If A is open and simply connected. Let $f: A \to \mathbb{C}$ be analytic and C^1. Then condition (ii) of Theorem 1 holds by Cauchy's Theorem, so we see that f has an antiderivative G given by

$$G(w) = \int_{\gamma} f(z)\, \mathrm{d}z$$

where we fix any $z_0 \in A$ and let γ be any piecewise C^1 curve in A connecting z_0 to w.

proof: Explain why the path-independence condition (ii) of Theorem 1 holds. (This explanation is somewhat imprecise, but we'll make everything rigorous in section 2.3)

Appendix: Connected domains, path connected domains, simply connected domains: Some Chapter 1 analysis background material we need now:

Recall that a domain $A \subseteq \mathbb{C}$ is called *connected* iff there is no disconnection of A into disjoint (relatively) open and non-empty subsets U, V i.e. such that

$$A = U \cup V$$
$$U \cap V = \varnothing.$$

If we restrict to open domains A, then subsets U, V that are relatively open are actually open.

There is a related definition:

<u>Def</u> A subset $A \subseteq \mathbb{C}$ is called *path connected* iff $\forall\ P, Q \in A$, there exists a continuous path $\gamma : [a, b] \rightarrow A$ such that $\gamma(a) = P, \gamma(b) = Q$.

<u>Theorem</u> Let $A \subseteq \mathbb{C}$ be open. Then A is connected if and only if A is path connected. Furthermore, if A is connected then there are piecewise C^1 paths connecting all possible pairs of points in A. (Analogous theorem holds in $\mathbb{R}^n$.)

<u>proof:</u> $\Rightarrow$: Let A be connected and open. We will show it is path connected, with piecewise C^1 paths. Pick any base point $z_0 \in A$. Define U to be the set of points that can be connected to z_0 with a piecewise C^1 path. U is non-empty since $\mathrm{D}(z_0; r) \subseteq U$ as long as r is small enough so that the disk is in A. In fact, for all $z \in \mathrm{D}(z_0; r)$ we can use the straight-line paths

$$\gamma(t) = z_0 + t(z - z_0), \quad 0 \leq t \leq 1$$

to connect z_0 to z.

The proof that U is open is analogous: Let $w \in U$ and let γ be a piecewise C^1 path connecting z_0 to w. Then for $\mathrm{D}(w, r) \subseteq A$ and

$$\gamma_1(t) = w + t(z - w), \quad 0 \leq t \leq 1$$

the combined path $\gamma + \gamma_1$ is a piecewise C^1 path connecting z_0 to w. Thus U is open.

Finally, the complement of $V := A \setminus U$ is open by a similar argument: Let $z_1 \in V$, $\mathrm{D}(z_1; r) \subseteq A$. Then $\mathrm{D}(z_1, r) \subseteq V$ as well, since if $\exists\ w \in U \cap \mathrm{D}(z_1; r)$ there is a piecewise C^1 path γ from z_0 to w, and letting

$$\gamma_2(t) = w + t(z_1 - w), \quad 0 \leq t \leq 1,$$

the path $\gamma + \gamma_2$ would connect z_0 to z_1. Thus, since A is connected, we must have that $V = A \setminus U$ is empty.

reverse implication: Assume A is path connected. Then A is connected: Let $A = U \cup V$ with U, V open, U non-empty, and $U \cap V = \varnothing$. We will show V is empty. If not, pick $P \in U, Q \in V$, and let

$$\gamma : [a, b] \to \mathbb{C}$$

be a continuous path connecting P to Q, i.e. $\gamma(a) = P, \gamma(b) = Q$. Let $T \in [a, b]$ be defined by

$$T := sup\{t \in [a, b] \mid \gamma([a, t]) \subseteq U\}$$

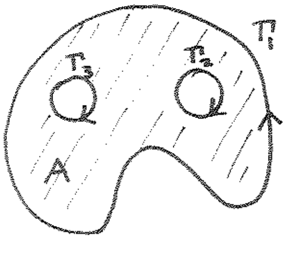
Because U is open, $T > a$. If $T = b$ we would have $Q \in U$ which would be a contradiction. But if $a < T < b$ then $\gamma(T)$ is in neither U nor V: If $\gamma(T) \in U$ then by continuity and U open, there exists $\delta > 0$ so that $\gamma([T, T + \delta) \subseteq U$, hence $\gamma([a, T + \delta) \subseteq U$, contradicting the definition of T. Similarly, if $\gamma(T) \in V$, continuity of γ and V open implies there exists $\delta > 0$ so that $\gamma([T - \delta, T] \subseteq V$, another contradiction. Thus T can't exist, and V must be empty.

Green's Theorem

(This is just one of the vector calculus "FTC"'s, and in fact one can understand all of them as special cases of a general theorem called Stokes' Theorem)

Let $\langle P(x,y), Q(x,y)\rangle$ be a vector field, C^1 on an open domain containing the set A and its boundary ∂A.

Orient $\Gamma = \partial A$ so that A is "on the left" as you traverse Γ:



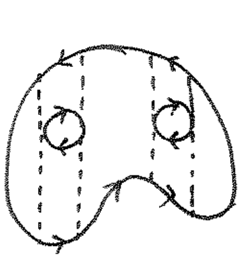
then
$$\oint_{\partial A} P\,dx + Q\,dy = \iint_A (Q_x - P_y)\,dA$$

proof: ① $\displaystyle\oint_{\partial A} P\,dx = \iint_A -P_y\,dA$, ② $\displaystyle\oint_{\partial A} Q\,dy = \iint_A Q_x\,dA$

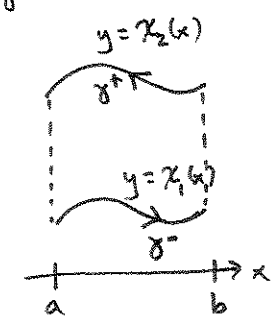
(unfortunately, I have used A for the region, and dA for dxdy — these uses of "A" are entirely independent)

① + ② = Green's Thm

① Chop up A into "vertical simple" subregions:



each region:



iterate the area integral:

$$\int_a^b \int_{\chi_1(x)}^{\chi_2(x)} -P_y(x,y)\,dy\,dx$$

$$= \int_a^b -P(x,y)\Big]_{y=\chi_1(x)}^{y=\chi_2(x)} dx$$

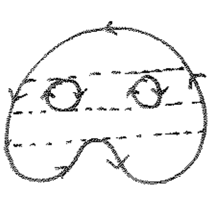
$$= \int_a^b -P(x,\chi_2(x))\,dx + P(x,\chi_1(x))\,dx$$

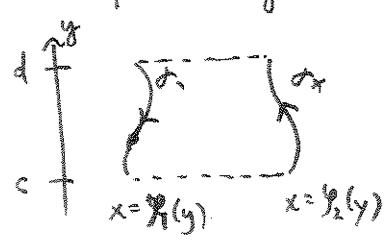
$$= \int_{\gamma^+} P\,dx + \int_{\gamma^-} P\,dx$$

add these identities for each region, and use the fact that double integrals and line integrals are additive with respect to partitioning.

Deduce $\displaystyle\iint_A -P_y\,dA = \oint_{\partial A} P\,dx$ ∎

② Chop A into "horizontal simple" subregions





$$\int_c^d \int_{\psi_1(y)}^{\psi_2(y)} Q_x\,dx\,dy = \int_c^d Q(x,y)\Big]_{x=\psi_1(y)}^{\psi_2(y)} dy$$

$$= \int_c^d Q(\psi_2(y),y)\,dy - Q(\psi_1(y),y)\,dy$$